D I S S E R T A T I O N

Clones on infinite sets

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Vorwort mit deutscher Kurzfassung

Sei X eine Menge. Eine fundamentale Frage des mathematischen Gebietes der universellen Algebra ist

Beschreibe die Menge aller universellen Algebren auf X.

Sei nun $\mathscr{A}$ eine universelle Algebra auf X. Viele Eigenschaften von $\mathscr{A}$, wie beispielsweise die Kongruenzen, die Unteralgebren, und die Automorphismen, hängen nicht von den fundamentalen Operationen von $\mathscr{A}$ ab, sondern von den Termoperationen, also jenen Operationen, welche von den fundamentalen Operationen und den Projektionen durch Funktionskomposition generiert werden. Aus diesem Grunde bezeichnen wir zwei universelle Algebren als *äquivalent* genau dann, wenn sie dieselben Termoperationen erzeugen. Modulo dieser Äquivalenz können wir obige Frage wie folgt formulieren:

Beschreibe die Menge aller Äquivalenzklassen von universellen Algebren auf X.

Ein *Klon* ist eine Menge von Termoperationen einer universellen Algebra auf X. Ebenso kann man einen Klon als Menge endlichstelliger Funktionen auf X, die alle Projektionen enthält und die unter Funktionskomposition abgeschlossen ist, definieren. Die Klone entsprechen also den Term-Äquivalenzklassen von universellen Algebren auf X. Ordnet man die Klone entsprechend der mengentheoretischen Inklusion, so erhält man einen vollständigen algebraischen Verband $Cl(X)$. Das Ziel der Klontheorie ist die Beantwortung eines bestimmten Aspektes obiger Frage, nämlich

Beschreibe $Cl(X)$.

Diese Dissertation behandelt Teile dieser Frage, hauptsächlich auf unendlichem X, und resultiert in einigen Struktursätzen über $Cl(X)$. Die Dissertation ist in eine Einleitung und drei Kapitel unterteilt, die unabhängig voneinander gelesen werden können. Die Kapitel entsprechen den Publikationen [Pin04a], [Pin04b], [Pin0x] des Autors.

Das Thema des ersten Kapitels sind Klone auf einer linear geordneten Grundmenge X, die endlich oder unendlich sein kann. Mithilfe der linearen Ordnung lassen sich gewisse natürliche Funktionen definieren, von denen wohl die natürlichsten die Maximum-, die Minimum-, und die *Medianfunktionen* sind, mit ihren offensichtlichen Definitionen. Während man leicht einsieht, daß eine Maximumfunktion mindestens zweier Veränderlicher auch die Maximumfunktionen anderer Stelligkeit erzeugt, und daß dasselbe für die Minimumfunktionen gilt, ist es nicht klar, ob beispielsweise der dreistellige Median die Medianfunktionen größerer Stelligkeit generiert. Unter Verwendung kombinatiorischer Methoden zeigen wir, daß dies tatsächlich der Fall, daß also alle Medianfunktionen denselben Klon generieren.

Das zweite Kapitel behandelt Klone auf unendlichen Grundmengen X regulärer Kardinalität. Eine Funktion heißt fast unär, falls eine ihrer Variablen den Funktionswert schon bis auf eine Menge bestimmt, deren Kardinalität kleiner als die von X ist. Die Menge aller fast unären Funktionen bildet einen Klon, der alle (echt) unären Funktionen enthält; dieser Klon spielt eine zentrale Rolle in der Struktur des Klonverbandes oberhalb der unären Funktionen. Wir bestimmen alle Klone, die den Klon der fast unären Funktionen enthalten. Es stellt sich heraus, daß diese Klone unabhängig von der Größe der Grundmenge eine abzählbar unendliche absteigende Kette bilden, deren Durchschnitt gerade der Klon der fast unären Funktionen ist.

Im dritten Kapitel wenden wir uns maximalen Klonen auf unendlichen Mengen zu. Dabei nennen wir einen Klon *maximal*, wenn er ein Dualatom des Klonverbandes ist. Es ist bekannt, daß die Menge der maximalen Klone auf unendlichem X schon so groß ist wie der gesamte Klonverband; daher gibt es wenig Hoffnung, alle maximalen Klone zu finden. Wir schränken die Menge der betrachteten Klone ein und erhalten auf unendlichem X regulärer Kardinalität eine explizite Liste aller maximalen Klone, die alle Permutationen, nicht aber alle unären Funktionen enthalten. Zudem bestimmen wir auf allen unendlichen Mengen X alle maximalen Submonoide des Transformationsmonoids, die die Permutationen von X enthalten.

CLONES ON INFINITE SETS

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Preface

Let X be a set. A fundamental problem of the field of universal algebra is

Describe the set of all universal algebras on X.

Consider a universal algebra $\mathscr{A}$ on X. Many properties of $\mathscr{A}$, such as its congruences, its subalgebras, and its automorphisms, do not depend on the fundamental operations of $\mathscr{A}$, but on its term operations, that is, the operations which are generated from its fundamental operations and the projections by function composition. We therefore call two universal algebras *equivalent* if and only if they have the same term operations. Up to this equivalence, we can reformulate our problem as follows:

Describe the set of all term equivalence classes of universal algebras on X.

A *clone* is a set of term operations of a universal algebra on X. Equivalently, a clone can be defined as a set of finitary operations on X which contains the projections and which is closed under composition. The set of all clones on X thus corresponds to the set of term equivalence classes of universal algebras on X. Ordering this set by set-theoretical inclusion, one obtains a complete algebraic lattice $Cl(X)$. The aim of clone theory is the solution of a certain aspect of the above-mentioned problem, namely

Describe $Cl(X)$.

This thesis treats instances of the latter question, mainly for infinite X, resulting in several structure theorems on $Cl(X)$. We divide this thesis into an introduction plus three chapters, all of which can be read independently. The chapters correspond to the author's publications [Pin04a], [Pin04b], [Pin0x].

The first chapter deals with clones on a linearly ordered base set X (finite or infinite). Using the linear order, certain natural functions can be defined, the most natural ones being the maximum, the minimum, and the *median functions*, with their obvious definitions. Whereas it is easily seen that any maximum function of at least two variables

generates the maximum functions of all arities, and that the same is true for the minimum functions, it is not clear that the median of, say, three variables generates the median functions of larger arities. Using combinatorial methods, we show that this is indeed the case, that is, all median functions generate the same clone.

In the second chapter, we turn to base sets X of infinite regular cardinality. A function is called *almost unary* iff one of its variables determines the value of the function up to a set of cardinality smaller than the cardinality of X. The set of all almost unary functions forms a clone which contains all (really) unary functions; this clone is of importance for the structure of the clone lattice above the unary functions. We determine all clones containing all almost unary functions; it turns out that independently of the size of X, these clones are a countably infinite descending chain with the almost unary functions as its intersection.

Chapter 3 is devoted to maximal clones on infinite sets. A clone is called *maximal* iff it is a dual atom in $Cl(X)$. Because the number of maximal clones on an infinite set equals the size of the whole clone lattice, there is little hope to find all of them. We restrict the set of clones under consideration and provide on all infinite X of regular cardinality an explicit list of all maximal clones which contain all permutations of X but not all unary functions. Moreover, we determine on all infinite X the maximal submonoids of the full transformation monoid which contain the permutations.

Contents

Introduction

Let X be a set of size $|X| = \kappa$ and denote by $\mathscr{O}^{(n)}$ the set of all n-ary functions on X. Then $\mathscr{O} = \bigcup_{n=1}^{\infty} \mathscr{O}^{(n)}$ is the set of all finitary functions on X. A *clone* $\mathscr{C}$ over X is a subset of $\mathscr{O}$ which contains the projections, i.e. the functions of the form $\pi_k^n(x_1, \ldots, x_n) = x_k$ $(1 \leq k \leq n)$, and which is closed under composition. Since arbitrary intersections of clones are obviously again clones, the set of all clones over X forms a complete lattice $Cl(X)$ with respect to inclusion. This lattice is a subset of the power set of $\mathscr{O}$. The clone lattice is countably infinite if X has only two elements, and has been completely determined in that case by E. Post [Pos41]. If X is finite and has at least three elements, $Cl(X)$ is already of size $2^{\aleph_0}$. For infinite X we have $|Cl(X)| = 2^{2^\kappa}$. Because the clone lattice is so large in the latter two cases, it is unlikely that it will ever be fully described. The approach of clone theory is to investigate interesting parts of the lattice, such as the maximal clones, the minimal clones, or natural intervals in the lattice.

A clone is called *maximal* iff it is a dual atom in $Cl(X)$. On finite X there exist finitely many maximal clones and an explicit list of those clones has been provided by I. Rosenberg [Ros70] (see also the diploma thesis [Pin02] for a self-contained proof of Rosenberg's Theorem). Moreover, the clone lattice is dually atomic in that case, that is, every clone is contained in a maximal one. If X is infinite, then the number of maximal clones equals the size of the whole clone lattice ([Ros76], see also [GS02]), so that it seems impossible to determine all of them. It has also been shown [GS04] that if the continuum hypothesis holds, then the clone lattice on a countably infinite base set is not dually atomic. We will deal with maximal clones in Chapters 2 and 3: In the second chapter, we obtain on all X of infinite regular cardinality a simple description of a certain maximal clone above $\mathscr{O}^{(1)}$ which is important for the structure of the interval $[\mathscr{O}^{(1)}, \mathscr{O}]$ of the clone lattice. In the third chapter, we give an explicit list of the maximal clones which contain the set $\mathscr{S}$ of all permutations on X but which do not contain $\mathscr{O}^{(1)}$.

A *minimal* clone on X is an atom in the lattice $Cl(X)$, i.e. a minimal element in

$Cl(X) \setminus \{\mathscr{J}\}$, where $\mathscr{J}$ is the trivial clone containing only the projections. Clearly every minimal clone is generated by a single nontrivial function. Functions which generate minimal clones are called *minimal* as well. On finite X, the minimal clones are finite in number and every clone contains a minimal one. Surprisingly, there is no characterization of minimal clones even on finite X. If we take the base set X to be infinite, then the number of minimal clones is 2^κ, and it is easy to see that not every clone contains a minimal one. The first chapter deals with a certain minimal clone on a linearly ordered base set X, namely the clone generated by the median functions.

Because the clone lattice is too large to completely understand it, it makes sense to pick feasible *intervals* of it and try to determine them. For example, there exist a number of results on the interval $[\mathscr{O}^{(1)}, \mathscr{O}]$ of clones containing all unary functions. One such result due to G. Gavrilov [Gav65] is that on countably infinite X, there exist only two maximal clones in this interval. M. Goldstern and S. Shelah [GS04] proved that the same is true on X of weakly compact cardinality, but showed in the same article that on most other cardinals, in particular on all successors of regulars, there exist 2^{2^κ} such clones. We will prove another structure theorem for clones above $\mathscr{O}^{(1)}$ in Chapter 2, determining the interval $[\mathscr{U}, \mathscr{O}]$ of clones containing all almost unary functions.
Another example of an interesting interval in the interval $[\mathscr{S}, \mathscr{O}]$ of clones containing all permutations of X. L. Heindorf [Hei02] determined on countably infinite X all maximal clones in this interval. We will extend his result to all infinite X of regular cardinality in Chapter 3, obtaining an explicit list of all maximal clones which contain the permutations but not $\mathscr{O}^{(1)}$.
The interval $[\mathscr{J}, \mathscr{O}^{(1)}]$ consists of those clones which contain only *essentially unary* functions, i.e. functions that depend only on one of their variables. Such clones are essentially submonoids of the full transformation monoid $\mathscr{O}^{(1)}$. It is known that the number of dual atoms in this interval is 2^{2^κ}, so there is no hope to determine them. However, G. Gavrilov [Gav65] found all dual atoms of this interval which contain $\mathscr{S}$ (so he found the dual atoms of $[\mathscr{S}, \mathscr{O}^{(1)}]$), on countably infinite X. We will generalize his theorem to all infinite X in Chapter 3.

For extensive introductions to clone theory (although primarily on finite base sets), we refer to the monograph [Sze86] by Á. Szendrei and the textbook [PK79] by R. Pöschel and L. Kalužnin.

Chapter 1

The clone generated by the median functions

Let X be a linearly ordered set of arbitrary size (finite or infinite). Natural functions on such a set one can define using the linear order include maximum, minimum and median functions. While it is clear what the clone generated by the maximum or the minimum looks like, this is not obvious for the median functions. We show that every clone on X contains either no median function or all median functions, that is, the median functions generate each other.

1.1 The median functions

Assume X to be linearly ordered. We emphasize that the cardinality of X is not relevant. For all $n \geq 1$ and all $1 \leq k \leq n$ we define a function

$$m_k^n(x_1, \ldots, x_n) = x_{j_k} \quad \text{if } x_{j_1} \leq \ldots \leq x_{j_n}.$$

In words, the function m_k^n returns the k-th smallest element from an n-tuple. The functions m_k^n are totally symmetric, i.e., invariant unter all permutations of their variables, and $m_k^n(x_1, \ldots, x_n) = x_k$ whenever $x_1 \leq \ldots \leq x_n$. For example, m_n^n is the maximum function $\max_n$ and m_1^n the minimum function $\min_n$ in n variables. If n is an odd number then we call $m_{\frac{n+1}{2}}^n$ the n-th median function and denote this function by med_n.

It is easy to check what the clones generated by the functions max and min look like:

$$\langle\{\max_n\}\rangle = \{\max_k(\pi_{i_1}^j, \ldots, \pi_{i_k}^j) : 1 \leq i_1, \ldots, i_k \leq j,\ 1 \leq k \leq j\}$$

and

$$\langle\{\min_n\}\rangle = \{\min_k(\pi^j_{i_1}, \ldots, \pi^j_{i_k}) : 1 \leq i_1, \ldots, i_k \leq j,\, 1 \leq k \leq j\},$$

where $n \geq 2$ is arbitrary. In particular, the two clones are minimal. Now it is natural to ask which of these properties hold for the functions “in between”, that is the m^n_k as defined before, most importantly the median functions. We will show that for odd $n \geq 3$

$$\langle\{\mathrm{med}_n\}\rangle \supseteq \{\mathrm{med}_k(\pi^j_{i_1}, \ldots, \pi^j_{i_k}) : 1 \leq i_1, \ldots, i_k \leq j,\, 1 \leq k \leq j,\, k \text{ odd}\},$$

but one readily constructs functions in that clone which are not a median function and not a projection. However, R. Pöschel and L. Kalužnin observed in [PK79], Theorem 4.4.5, that the median of three variables (and hence by our result, all medians) does generate a minimal clone.

Theorem 1. *The clone generated by the function* med_3 *is minimal.*

We are going to prove

Theorem 2. *Let $k, n \geq 3$ be odd natural numbers. Then* $\mathrm{med}_k \in \langle\{\mathrm{med}_n\}\rangle$*. In other words, a clone contains either no median function or all median functions.*

1.1.1 Notation

For a set of functions $\mathscr{F}$ we shall denote the smallest clone containing $\mathscr{F}$ by $\langle\mathscr{F}\rangle$. If $1 \leq k \leq n$, we write π^n_k for the n-ary projection on the k-th component.
For a positive rational number q we write

$$\lfloor q \rfloor = \max\{n \in \mathbb{N} :\, n \leq q\}$$

and

$$\lceil q \rceil = \min\{n \in \mathbb{N} :\, q \leq n\}.$$

If $a \in X^n$ is an n-tuple and $1 \leq k \leq n$ we write a_k for the k-th component of a. We will assume X to be linearly ordered by the relation $\leq$ and let $<$ carry the obvious meaning.

1.2 The proof of Theorem 2

1.2.1 Almost divisibility

We split the proof of the theorem into a sequence of lemmas.

Definition 3. Let $k, n \geq 1$ be natural numbers. Denote by $R(\frac{n}{k})$ the remainder of the division $\frac{n}{k}$. We say that n is *almost divisible by* k iff either $R(\frac{n}{k}) \leq \frac{n}{k}$ or $(k-1) - R(\frac{n}{k}) \leq \frac{n}{k}$.

Note that n is almost divisible by k if it is divisible by k. The following lemma tells us which medians of smaller arity are generated by med_n by simple identification of variables (see also Remark 13).

Lemma 4. *Let $k \leq n$ be odd natural numbers. If n is almost divisible by k, then* $\mathrm{med}_k \in \langle \{\mathrm{med}_n\} \rangle$.

Proof. We claim that

$$\mathrm{med}_k(x_1, \ldots, x_k) = \mathrm{med}_n(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_k, \ldots, x_k),$$

where x_j occurs in the n-tuple $\lfloor \frac{n}{k} \rfloor + 1$ times if $j \leq R(\frac{n}{k})$ and $\lfloor \frac{n}{k} \rfloor$ times otherwise. Assume $\mathrm{med}_k(x_1, \ldots, x_k) = x_j$. Then there are at most $\frac{k-1}{2}$ components smaller than x_j and at most $\frac{k-1}{2}$ components larger than x_j. Thus in our n-tuple, there are at most

$$\frac{k-1}{2} \lfloor \frac{n}{k} \rfloor + \min(R(\frac{n}{k}), \frac{k-1}{2}) \tag{1.1}$$

elements smaller (larger) than x_j.
Case 1. $R(\frac{n}{k}) \leq \frac{k-1}{2}$.
Since n is almost divisible by k, we have either $R(\frac{n}{k}) \leq \frac{n}{k}$ or $(k-1) - R(\frac{n}{k}) \leq \frac{n}{k}$. In the latter case,

$$R(\frac{n}{k}) \leq \frac{k-1}{2} \quad \wedge \quad (k-1) - R(\frac{n}{k}) \leq \frac{n}{k}$$

and so

$$R(\frac{n}{k}) \leq \frac{n}{k}.$$

Thus in either of the cases, we can calculate from (1.1)

$$\begin{aligned}
&\frac{k-1}{2} \lfloor \frac{n}{k} \rfloor + R(\frac{n}{k}) \\
&= \frac{1}{2}(k \lfloor \frac{n}{k} \rfloor + R(\frac{n}{k})) + \frac{1}{2}(R(\frac{n}{k}) - \lfloor \frac{n}{k} \rfloor) \\
&= \frac{n}{2} + \frac{1}{2}(R(\frac{n}{k}) - \lfloor \frac{n}{k} \rfloor) \\
&\leq \frac{n}{2}
\end{aligned}$$

and so med_n yields x_j.
Case 2. $\frac{k-1}{2} < R(\frac{n}{k})$.

Again we know that either $R(\frac{n}{k}) \leq \frac{n}{k}$ or $(k-1) - R(\frac{n}{k}) \leq \frac{n}{k}$. In the first case, we see that

$$\frac{k-1}{2} < R(\frac{n}{k}) \quad \wedge \quad R(\frac{n}{k}) \leq \frac{n}{k}$$

implies

$$(k-1) - R(\frac{n}{k}) \leq \frac{n}{k}$$

and so (1.1) yields at most

$$\begin{aligned}
&\frac{k-1}{2}\lfloor\frac{n}{k}\rfloor + \frac{k-1}{2} \\
&= \frac{k-1}{2}\lfloor\frac{n}{k}\rfloor + \frac{1}{2}R(\frac{n}{k}) + \frac{k-1}{2} - \frac{1}{2}R(\frac{n}{k}) \\
&= \frac{1}{2}(k\lfloor\frac{n}{k}\rfloor + R(\frac{n}{k})) - \frac{1}{2}\lfloor\frac{n}{k}\rfloor + \frac{k-1}{2} - \frac{1}{2}R(\frac{n}{k}) \\
&\leq \frac{n}{2} + \frac{1}{2}(-\lfloor\frac{n}{k}\rfloor + (k-1) - R(\frac{n}{k})) \\
&\leq \frac{n}{2}
\end{aligned}$$

components which are smaller (larger) than x_j. This finishes the proof. □

Corollary 5. *Let $k, n \geq 1$ be odd natural numbers. If $k \leq \sqrt{n}$, then* med_k *is generated by* med_n.

Proof. Trivially, $R(\frac{n}{k}) \leq k-1$ and $k-1 \leq \frac{n}{k}$ as $k \leq \sqrt{n}$. Hence, n is almost divisible by k. □

Corollary 6. *Let $n \geq 3$ be odd. Then* $\mathrm{med}_3 \in \langle\{\mathrm{med}_n\}\rangle$.

Proof. Simply observe that all $n \geq 4$ are almost divisible by 3. □

1.2.2 Majority functions

We have seen that we can get small (that is, of small arity) median functions out of large ones. The converse inclusion is shown with the help of majority functions.

Definition 7. Let $f \in \mathscr{O}^{(n)}$. We say that f is a *majority function* iff $f(x_1, \ldots, x_n) = x$ whenever the value x occurs at least $\lceil\frac{n+1}{2}\rceil$ times among $(x_1, \ldots, x_n)$.

Note that med_n is a majority function for all odd n. We observe now that we can build a ternary majority function from most larger ones by identifying variables.

Lemma 8. *Let $n \geq 5$ and let $\mathrm{maj}_n \in \mathscr{O}^{(n)}$ be a majority function. Then maj_n generates a majority function of three arguments.*

Proof. Set
$$\mathrm{maj}_3 = \mathrm{maj}_n(x_1, \ldots, x_1, x_2, \ldots, x_2, x_3, \ldots, x_3),$$
where x_j occurs in the n-tuple $\lfloor \frac{n}{3} \rfloor + 1$ times if $j \leq R(\frac{n}{3})$ and $\lfloor \frac{n}{3} \rfloor$ times otherwise. It is readily verified that maj_3 is a majority function. □

The following lemma tells us that we can generate majority functions of even arity from majority functions of odd arity.

Lemma 9. *Let $n \geq 2$ be an even natural number. Then we can get an n-ary majority function maj_n out of any $(n+1)$-ary majority function maj_{n+1}.*

Proof. Set
$$\mathrm{maj}_n(x_1, \ldots, x_n) = \mathrm{maj}_{n+1}(x_1, \ldots, x_n, x_n)$$
and let $x \in X$ have a majority among $(x_1, \ldots, x_n)$. Since n is even, x occurs $\frac{n}{2}+1$ times in the n-tuple which is enough for a majority in the $(n+1)$-tuple $(x_1, \ldots, x_n, x_n)$. □

We now show that we can construct large majority functions out of small ones. This has already been known but we include our own proof here.

Lemma 10. *Let $n \geq 5$ be a natural number. Then we can construct an n-ary majority function out of any $(n-2)$-ary majority function maj_{n-2}.*

Proof. For $2 \leq j \leq n-1$ and $1 \leq i \leq n-1$ with $i \neq j$ we define functions
$$\gamma_i^j = \begin{cases} \mathrm{maj}_{n-2}(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n) & j \neq i+1 \\ \mathrm{maj}_{n-2}(x_1, \ldots, x_{i-1}, x_{i+1}, x_{i+3}, \ldots, x_n) & j = i+1 \end{cases}$$
In words, given an n-tuple $(x_1, \ldots, x_n)$, γ_i^j ignores x_i and the next component of the n-tuple which is not x_j and calculates maj_{n-2} from what is left. Set
$$z_j = \mathrm{maj}_{n-2}(\gamma_1^j, \ldots, \gamma_{j-1}^j, \gamma_{j+1}^j, \ldots, \gamma_{n-1}^j)$$
and
$$f = \mathrm{maj}_{n-2}(z_2, \ldots, z_{n-1}).$$
The function f is an n-ary term of depth three over $\{\mathrm{maj}_{n-2}\}$.
Claim. f is a majority function.
We prove our claim for the case where n is odd. The same proof works in the even

case, the only difference being that the counting is slightly different (a majority occurs $\frac{n+2}{2}$ times instead of $\frac{n+1}{2}$, and so on). We leave the verification of this to the diligent reader.

Assume $x \in X$ has a majority. If x occurs more than $\frac{n+1}{2}$ times, then it is readily verified that all the γ_i^j yield x and so do all z_j and so does f. So say x appears exactly $\frac{n+1}{2}$ times among the variables of f.

Next we observe that if $x_j = x$, then $z_j = x$: For if $\gamma_i^j \neq x$, then both components ignored in γ_i^j, that is, x_i and the component after x_i which is not x_j, have to be equal to x. We can count

$$|\{i : \gamma_i^j \neq x\}| \leq |\{i \neq j : x_i = x\} \setminus \{\max(i \neq j : x_i = x)\}| \leq \frac{n-1}{2} - 1 = \frac{n-3}{2}.$$

Thus, $z_j = x$.

Now we shall count a second time to see that if $x_1 \neq x$ or $x_n \neq x$, then $f = x$: Say without loss of generality $x_1 \neq x$. Then

$$|\{2 \leq j \leq n-1 : x_j = x\}| \geq \frac{n+1}{2} - 1 = \frac{n-1}{2}$$

and since we have seen that $z_j = x$ for all such j we indeed obtain $f = x$.

In a last step we consider the case where both $x_1 = x$ and $x_n = x$. Let

$$k = \min\{i : x_i \neq x\}$$

and

$$l = \max\{i : x_i \neq x\}.$$

Since $n \geq 5$ those two indices are not equal. Count

$$|\{i : \gamma_i^l \neq x\}| \leq |\{i : x_i = x\} \setminus \{k-1, n\}| = \frac{n+1}{2} - 2 = \frac{n-3}{2}.$$

Thus, $z_l = x$ and we count for the last time

$$|\{j : z_j = x\}| \geq |\{2 \leq j \leq n-1 : x_j = x\} \cup \{l\}| = \frac{n-3}{2} + 1 = \frac{n-1}{2},$$

so that also in this case $f = x$. $\square$

We conclude that if a clone contains a majority function, then it contains majority functions of all arities.

Corollary 11. *Let $n, k \geq 3$ be natural numbers. Assume $\mathrm{maj}_n \in \mathscr{O}^{(n)}$ is any majority function. Then maj_n generates a majority function in $\mathscr{O}^{(k)}$.*

Proof. If $k \geq n$ and n, k are either both even or both odd, then we can iterate Lemma 10 to generate a majority function of arity k. Lemma 9 takes care of the case when k is even but n is odd.

In all other cases with $n \geq 5$, generate a ternary majority function from maj_n first with the help of Lemma 8 and follow the procedure just described for the other case.

Finally, if $n = 4$, we can build a majority function maj_6 from maj_4 first and are back in one of the other cases. □

Now we use the large majority functions to obtain large median functions.

Lemma 12. *For all odd $n \geq 3$ there exists $b \geq n$ such that $\mathrm{med}_n \in \langle\{\mathrm{med}_3, \mathrm{maj}_b\}\rangle$ for an arbitrary b-ary majority function maj_b.*

Proof. Let n be given. Our strategy to calculate the median from an n-tuple will be the following: We apply med_3 to all possible selections of three elements of the n-tuple. The results we write to an n_1-tuple, from which we again take all possible selections of three elements. We apply med_3 again to these selections and so forth. Now the true median of the original n-tuple "wins" much more often in this procedure than the other elements, so that after a finite number of steps (a number we can give a bound for) more than half of the components of the then giant tuple have the true median as their value. To that tuple we apply a majority function and obtain the median.

In detail, we define two sequences $(n_j)_{j\in\omega}$ and $(k_j)_{j\in\omega}$ by

$$n_0 = n, \quad n_{j+1} = \binom{n_j}{3}$$

and

$$k_0 = 1, \quad k_{j+1} = \binom{k_j}{3} + \binom{k_j}{2}(n_j - k_j) + \binom{k_j}{1}(\frac{n_j - k_j}{2})^2.$$

The sequences have the following meaning: Given an n_j-tuple, there are n_{j+1} possible selections of three elements of the tuple to which we apply the median med_3. If the median of the n_j-tuple (which is equal to the median of the $n_0 = n$-tuple) appeared at least k_j times there, then it appears at least k_{j+1} times in the resulting n_{j+1}-tuple. Read k_{j+1} as follows: We assume the worst case, namely that the median occurs only once in the original n-tuple, so $k_0 = 1$. If we pick three elements from the n_j-tuple and calculate med_3, then the result is the median we are looking for if either all three elements are equal to the median ($\binom{k_j}{3}$ possibilities) or two are equal to the median ($\binom{k_j}{2}(n_j - k_j)$ possibilities) or one is equal to the median, one is smaller, and one is

larger ($\binom{k_j}{1}\,(\frac{n_j-k_j}{2})^2$ possibilities). Set $r_j = \frac{k_j}{n_j}$ for $j \geq 0$ to be the relative frequency of the median in the tuple after j steps. We claim that $\limsup(r_j)_{j\in\omega} = 1$:

$$\begin{aligned} r_{j+1} &= \frac{k_{j+1}}{n_{j+1}} \\ &= \frac{k_j}{n_j}\ \frac{(k_j-1)(k_j-2)+3(k_j-1)(n_j-k_j)+\frac{3}{2}(n_j^2-2n_jk_j+k_j^2)}{(n_j-1)(n_j-2)} \\ &= r_j\ \frac{3(n_j-1)^2+1-k_j^2}{2(n_j-1)(n_j-2)} \end{aligned}$$

Further calculation yields

$$\begin{aligned} r_{j+1} &\geq r_j\ \frac{3(n_j-1)^2+1-k_j^2}{2(n_j-1)^2} \\ &= r_j\ (\frac{3}{2}-\frac{(k_j-1)^2}{2(n_j-1)^2}-\frac{k_j-1}{(n_j-1)^2}) \\ &\geq r_j\ (\frac{3}{2}-\frac{1}{2}r_j^2-\frac{r_j}{n_j-1}) \\ &\geq r_j\ (\frac{3}{2}-\frac{1}{2}r_j^2-\frac{1}{n_j-1}). \end{aligned}$$

Suppose towards a contradiction that $(r_j)_{j\in\omega}$ is bounded away from 1 by p: $r_j < p < 1$ for all $j \in \omega$. Choose j large enough so that

$$\frac{1}{n_j-1} < \frac{1-p}{4}.$$

Then

$$\begin{aligned} r_{i+1} &> r_i\ (\frac{3}{2}-\frac{p}{2}-\frac{1-p}{4}) \\ &= r_i\ (1+\frac{1-p}{4}) \end{aligned}$$

for all $i \geq j$ so that there exists $l > j$ such that $r_l > p$, in contradiction to our assumption. Hence, $\limsup(r_j)_{j\in\omega} = 1$.

Now if we calculate j such that $r_j > \frac{1}{2}$, and choose $b = n_j$, we can obtain the median with the help of a b-ary majority function. □

We are ready to prove our main theorem.

Proof of Theorem 2. Let k, n be given. Corollary 6 tells us that we can construct med_3 out of med_n. Since med_3 is also a majority function, we can get majority functions of arbitrary arity with the help of Corollary 11. Then by the preceding lemma, we can generate med_k. $\square$

Remark 13. In fact, the lemma on almost divisibility is not needed for the proof of the theorem, since we only have to get med_3 out of med_n (and $\mathrm{med}_3(x_1, x_2, x_3) = \mathrm{med}_n(x_1, x_2, \ldots, x_2, x_3, \ldots, x_3)$ where x_2 and x_3 occur $\frac{n-1}{2}$ times in the n-tuple) and then apply Lemma 10 to generate large majority functions. Still, the lemma shows what we can construct by simple identification of variables.

1.3 Minimality of the m_k^n

We mentioned that the clones generated by the maximum, the minimum and the median functions are minimal. Anyone who hoped that the same holds for all m_k^n will be disappointed by the following lemma.

Lemma 14. *Let $n \geq 4$ and $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Then m_k^n is not a minimal function.*

Proof. It is enough to see that

$$\min_2(x, y) = m_k^n(x, \ldots, x, y, \ldots, y) \in \langle \{m_k^n\} \rangle,$$

where x occurs in the n-tuple exactly $\lfloor \frac{n}{2} \rfloor$ times. The clone generated by $\min_2$ is obviously a nontrivial proper subclone of $\langle \{m_k^n\} \rangle$. $\square$

Now comes just another disappointment.

Lemma 15. *Let $\lceil \frac{n}{2} \rceil < k < n$. Then m_k^n is not a minimal function.*

Proof. This time we have that

$$\max_2(x, y) = m_k^n(x, \ldots, x, y, \ldots, y) \in \langle \{m_k^n\} \rangle,$$

where x occurs in the n-tuple exactly $\lfloor \frac{n}{2} \rfloor$ times. The clone generated by the maximum functions is obviously a proper subclone of $\langle \{m_k^n\} \rangle$. $\square$

We summarize our results in the following corollary.

Corollary 16. *Let $n \geq 2$ and $1 \leq k \leq n$. Then m_k^n is minimal iff $k = 1$ or $k = n$ or n is odd and $k = \frac{n+1}{2}$. That is, the minimal functions among the m_k^n are exactly the maximum, the minimum and the median functions.*

1.4 Variations of the median function

For even natural numbers n we did not define median functions. One could consider the so-called "lower median" instead:

$$\mathrm{med}_n^{low} = m_{\frac{n}{2}}^n$$

But as a consequence of the preceding corollary, med_n^{low} is not generated by the real medians and does therefore not serve as a perfect substitute. For the same reason, the "upper median"

$$\mathrm{med}_n^{upp} = m_{\frac{n}{2}+1}^n$$

is not an ideal replacement either.

However, the other direction almost works: med_n^{low} generates the medians if and only if $n \geq 6$. Indeed, simple identification of variables suffices:

$$\mathrm{med}_3 = \mathrm{med}_n^{low}(x_1, \ldots, x_1, x_2, \ldots, x_2, x_3, \ldots, x_3)$$

where x_j occurs in the n-tuple $\lfloor \frac{n}{3} \rfloor + 1$ times if $j \leq R(\frac{n}{3})$ and $\lfloor \frac{n}{3} \rfloor$ times otherwise. Of course we can do the same with the upper medians. It is easy to see that med_4^{low} cannot generate the medians.

One could have the idea of using a more general notion of median functions: Let $(X, \wedge, \vee)$ be a lattice. Define

$$\widetilde{m}_k^n(x_1, \ldots, x_n) = \bigwedge_{(j_1,\ldots,j_k)\in\{1,\ldots,n\}^k} \bigvee_{1\leq i\leq k} x_{j_i}.$$

If the order induced by the lattice on X is a chain, this definition agrees with our definition of m_k^n. However, although we can get $\widetilde{\mathrm{med}}_3$ out of $\widetilde{\mathrm{med}}_n$ just like described in Remark 13, our proof to obtain large medians via majority functions fails. We do not know under which conditions on the lattice the same results can be obtained.

Chapter 2

Clones containing all almost unary functions

Let X be an infinite set of regular cardinality. We determine all clones on X which contain all almost unary functions. It turns out that independently of the size of X, these clones form a countably infinite descending chain. Moreover, all such clones are finitely generated over the unary functions. In particular, we obtain an explicit description of the only maximal clone in this part of the clone lattice. This is especially interesting if X is countably infinite, in which case it is known that such a simple description cannot be obtained for the second maximal clone over the unary functions.

2.1 Background

2.1.1 Almost unary functions

Let X be of infinite regular cardinality from now on unless otherwise stated. We call a subset $S \subseteq X$ *large* iff $|S| = |X|$, and *small* otherwise. If X is itself a regular cardinal, then the small subsets are exactly the bounded subsets of X. A function $f(x_1, \ldots, x_n) \in \mathscr{O}^{(n)}$ is *almost unary* iff there exists a function $F : X \to \mathscr{P}(X)$ and $1 \leq k \leq n$ such that $F(x)$ is small for all $x \in X$ and such that for all $(x_1, \ldots, x_n) \in X^n$ we have $f(x_1, \ldots, x_n) \in F(x_k)$. If we assume X to be a regular cardinal itself, this is equivalent to the existence of a function $F \in \mathscr{O}^{(1)}$ and a $1 \leq k \leq n$ such that $f(x_1, \ldots, x_n) < F(x_k)$ for all $(x_1, \ldots, x_n) \in X^n$. Because this is much more convenient and does not influence the properties of the clone lattice, we shall assume X to be a regular cardinal throughout this chapter. Let $\mathscr{U}$ be the set of all almost unary

functions. It is readily verified that $\mathscr{U}$ is a clone. We will determine all clones which contain $\mathscr{U}$; in particular, such clones contain $\mathscr{O}^{(1)}$.

2.1.2 Maximal clones above $\mathscr{O}^{(1)}$

Although on an infinite set X not every clone must be contained in a maximal one [GS04], the sublattice of $Cl(X)$ of functions containing $\mathscr{O}^{(1)}$ is dually atomic by Zorn's lemma, since $\mathscr{O}$ is finitely generated over $\mathscr{O}^{(1)}$. G. Gavrilov proved in [Gav65] that for countably infinite X there are only two maximal clones containing all unary functions. M. Goldstern and S. Shelah extended this result to clones on weakly compact cardinals in the article [GS02], where an uncountable cardinal X is called *weakly compact* iff whenever we colour the edges of a complete graph G of size X with two colours, then there exists a complete subgraph of G of size X on which the colouring is constant. In the same paper, the authors proved that on other regular cardinals X satisfying a certain partition relation there are even 2^{2^X} maximal clones above $\mathscr{O}^{(1)}$.

There exists exactly one maximal clone above $\mathscr{U}$. So far, this clone has been defined using the following concept: Let $\rho \subseteq X^J$ be a relation on X indexed by J and let $f \in \mathscr{O}^{(n)}$. We say that f *preserves* ρ iff for all $r^1 = (r^1_i : i \in J), \ldots, r^n = (r^n_i : i \in J)$ in ρ we have $(f(r^1_i, \ldots, r^n_i) : i \in J) \in \rho$. We define the set of *polymorphisms* $\mathrm{Pol}(\rho)$ of ρ to be the set of all functions in $\mathscr{O}$ preserving ρ; $\mathrm{Pol}(\rho)$ is easily seen to be a clone. In particular, if $\rho \subseteq X^{X^k}$ is a set of k-ary functions, then a function $f \in \mathscr{O}^{(n)}$ preserves ρ iff for all functions $g_1, \ldots, g_n$ in ρ the composite $f(g_1, \ldots, g_n)$ is a function in ρ.

Write
$$T_1 = \mathscr{U}^{(2)} = \{f \in \mathscr{O}^{(2)} : f \text{ almost unary}\}.$$

The following was observed by G. Gavrilov [Gav65] for countable base sets and extended to all regular X by R. Davies and I. Rosenberg [DR85]. Uniqueness on uncountable regular cardinals is due to M. Goldstern and S. Shelah [GS02].

Theorem 17. *Let X have infinite regular cardinality. Then* $\mathrm{Pol}(T_1)$ *is a maximal clone containing all unary functions. Furthermore,* $\mathrm{Pol}(T_1)$ *is the only maximal clone containing all almost unary functions.*

For S a subset of X we set

$$\Delta_S = \{(x,y) \in S^2 : y < x\}, \quad \nabla_S = \{(x,y) \in S^2 : x < y\}.$$

We will also write Δ and ∇ instead of Δ_X and ∇_X. Now define

$$T_2 = \{f \in \mathscr{O}^{(2)} : \forall S \subseteq X\, (S \text{ large} \rightarrow \text{ neither } f\restriction_{\Delta_S} \text{ nor } f\restriction_{\nabla_S} \text{ are 1-1})\}.$$

The next result is due to G. Gavrilov [Gav65] for X a countable set and due to M. Goldstern and S. Shelah [GS02] for X weakly compact.

Theorem 18. *Let X be countably infinite or weakly compact. Then* $\operatorname{Pol}(T_2)$ *is a maximal clone which contains $\mathscr{O}^{(1)}$. Moreover,* $\operatorname{Pol}(T_1)$ *and* $\operatorname{Pol}(T_2)$ *are the only maximal clones above $\mathscr{O}^{(1)}$.*

The definition of $\operatorname{Pol}(T_2)$ not only looks more complicated than the one of $\operatorname{Pol}(T_1)$. First of all, a result of R. Davies and I. Rosenberg in [DR85] shows that assuming the continuum hypothesis, T_2 is not closed under composition on $X = \aleph_1$ and so it is unclear what $\operatorname{Pol}(T_2)$ is. Secondly, on countable X, if we equip $\mathscr{O}$ with the natural topology which we shall specify later, then T_2 is a non-analytic set in that space and so is $\operatorname{Pol}(T_2)$; in particular, neither $\langle T_2\rangle$ nor $\operatorname{Pol}(T_2)$ are countably generated over $\mathscr{O}^{(1)}$ (see [Gol0x]), where for a set of functions $\mathscr{F}$ we denote by $\langle\mathscr{F}\rangle$ the clone generated by $\mathscr{F}$. The clones $\langle T_1\rangle$ and $\operatorname{Pol}(T_1)$ on the other hand turn out to be rather simple with respect to this topology, and both clones are finitely generated $\mathscr{O}^{(1)}$.

Fix any injection p from X^2 to X; for technical reasons we assume that $0 \in X$ is not in the range of p.

Fact 19. $\langle\{p\} \cup \mathscr{O}^{(1)}\rangle = \mathscr{O}$, *that is, the function p together with $\mathscr{O}^{(1)}$ generate $\mathscr{O}$.*

For a subset S of X^2 we write

$$p_S(x_1,x_2) = \begin{cases} p(x_1,x_2) & ,(x_1,x_2) \in S \\ 0 & ,\text{otherwise} \end{cases}$$

M. Goldstern observed the following [Gol0x]. Since the result has not yet been published, but is important for our investigations, we include a proof here.

Fact 20. $\langle\{p_\Delta\} \cup \mathscr{O}^{(1)}\rangle = \langle T_1\rangle$.

Proof. Set $\mathscr{C} = \langle\{p_\Delta\} \cup \mathscr{O}^{(1)}\rangle$. Since $p_\Delta(x_1,x_2)$ is obviously bounded by the unary function $\gamma(x_1) = \sup\{p_\Delta(x_1,x_2) : x_2 \in X\} + 1 = \sup\{p(x_1,x_2) : x_2 < x_1\} + 1$, where by $\alpha + 1$ we mean the successor of an ordinal α, we have $p_\Delta \in T_1$ and hence $\mathscr{C} \subseteq \langle T_1\rangle$.

To see the other inclusion, note first that the function

$$q(x_1,x_2) = \begin{cases} p_\Delta(x_1,x_2) & ,(x_1,x_2) \in \Delta \\ x_1 & ,\text{otherwise} \end{cases}$$

is in $\mathscr{C}$. Indeed, choose $\epsilon \in \mathscr{O}^{(1)}$ strictly increasing such that $p_\Delta(x_1,x_2) < \epsilon(x_1)$ for all $x_1,x_2 \in X$ and consider $t(x_1,x_2) = p_\Delta(\epsilon(x_1), p_\Delta(x_1,x_2))$. On Δ, t is still one-one,

and outside Δ, the term is a one-one function of the first component x_1. Moreover, the ranges $t[\Delta]$ and $t[X^2 \setminus \Delta]$ are disjoint. Hence, we can write $q = u \circ t$ for some unary u. By the same argument we see that for arbitrary unary functions $a, b \in \mathscr{O}^{(1)}$ the function

$$q_{a,b}(x_1, x_2) = \begin{cases} a(p_\Delta(x_1, x_2)) & , (x_1, x_2) \in \Delta \\ b(x_1) & , \text{otherwise} \end{cases}$$

is an element of $\mathscr{C}$.

Now let $f \in T_1$ be given and say $f(x_1, x_2) < \delta(x_1)$ for all $x_1, x_2 \in X$, where $\delta \in \mathscr{O}^{(1)}$ is strictly increasing. Choose $a \in \mathscr{O}^{(1)}$ such that $a(p_\Delta(x_1, x_2)) = f(x_1, x_2) + 1$ for all $(x_1, x_2) \in \Delta$. Then set

$$f_1(x_1, x_2) = q_{a,\delta+1}(x_1, x_2) = \begin{cases} f(x_1, x_2) + 1 & , (x_1, x_2) \in \Delta \\ \delta(x_1) + 1 & , \text{otherwise} \end{cases}$$

We construct a second function

$$f_2(x_1, x_2) = \begin{cases} 0 & , (x_1, x_2) \in \Delta \\ f(x_1, x_2) + 1 & , \text{otherwise} \end{cases}$$

It is readily verified that $f_2(x_1, x_2) = u(p_\Delta(x_2 + 1, x_1))$ for some unary u. Now $f_2(x_1, x_2) < f_1(x_1, x_2)$ and $f_1, f_2 \in \mathscr{C}$. Clearly

$$f(x_1, x_2) = u(p_\Delta(f_1(x_1, x_2), f_2(x_1, x_2)))$$

for some unary u. This shows $f \in \mathscr{C}$ and so $\langle T_1 \rangle \subseteq \mathscr{C}$ as $f \in T_1$ was arbitrary. $\square$

We shall see that $\mathrm{Pol}(T_1)$ is also finitely generated over $\mathscr{O}^{(1)}$. Moreover, for countable X it is a Borel set in the topology yet to be defined. Our explicit description $\mathrm{Pol}(T_1)$ holds for all infinite X of regular cardinality, but is interesting only if there are not too many other maximal clones containing $\mathscr{O}^{(1)}$. By Theorem 18, this is at least the case for X countably infinite or weakly compact.

Throughout this chapter, the assumption that the base set X has regular cardinality is essential. To give an example, we prove now that $\mathscr{U}$ is a clone. Let $f \in \mathscr{U}^{(n)}$ and $g_1, \ldots, g_n \in \mathscr{U}^{(m)}$. By definition, there exists $F \in \mathscr{O}^{(1)}$ and some $1 \leq k \leq n$ such that $f(x) < F(x_k)$ for all $x \in X^n$. Because $g_k \in \mathscr{U}$, we obtain $G_k \in \mathscr{O}^{(1)}$ and $1 \leq i \leq m$ such that $g_k(x) < G_k(x_i)$ for all $x \in X^m$. Therefore $f(g_1, \ldots, g_n)(x) < H(x_i)$, where we define $H(x_i) = \sup_{y < G_k(x_i)}\{F(y)\}$. Now since $F(y) < X$ for all $y \in X$, and since the supremum ranges over a set of size $G_k(x_i) < X$, the regularity of X implies that $H(x_i) < X$, so that the composite $f(g_1, \ldots, g_n)$ is bounded by a unary function and

hence an element of $\mathscr{U}$. It is easy to see that on singular X, neither of the definition of an almost unary function by means of small sets nor the one via boundedness by a unary function yield a clone. Also, the two definitions differ on singulars, whereas on regulars they coincide.

2.1.3 Notation

For a set of functions $\mathscr{F}$ we shall denote the smallest clone containing $\mathscr{F}$ by $\langle \mathscr{F} \rangle$. By $\mathscr{F}^{(n)}$ we refer to the set of n-ary functions in $\mathscr{F}$.
We call the projections which every clone contains π_i^n where $n \geq 1$ and $1 \leq i \leq n$. If $f \in \mathscr{O}^{(n)}$ is an n-ary function, it sends n-tuples of elements of X to X and we write $(x_1, \ldots, x_n)$ for these tuples unless otherwise stated as in $f(x, y, z)$; this is the only place where we do not stick to set-theoretical notation (according to which we would have to write $(x_0, \ldots, x_{n-1})$). The set $\{1, \ldots, n\}$ of indices of n-tuples will play an important role and we write N for it. We denote the set-theoretical complement of a subset $A \subseteq N$ in N by $-A$. We identify the set X^n of n-tuples with the set of functions from N to X, so that if $A \subseteq N$ and $a : A \to X$ and $b : -A \to X$ are partial functions, then $a \cup b$ is an n-tuple. Sometimes, if the arity of $f \in \mathscr{O}$ has not yet been given a name, we refer to that arity by n_f.
If $a \in X^n$ is an n-tuple and $1 \leq k \leq n$ we write $(a)_k^n$ or only a_k for the k-th component of a. For $c \in X$ and J an index set we write c^J for the J-tuple with constant value c. The order relation $\leq$ on X induces the pointwise partial order on the set of J-tuples of elements of X for any index set J: For $x, y \in X^J$ we write $x \leq y$ iff $x_j \leq y_j$ for all $j \in J$. Consequently we also denote the induced pointwise partial order of $\mathscr{O}^{(n)}$ by $\leq$, so that for $f, g \in \mathscr{O}^{(n)}$ we have $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X^n$. Whenever we state that a function $f \in \mathscr{O}^{(n)}$ is monotone, we mean it is monotone with respect to $\leq$: $f(x) \leq f(y)$ whenever $x \leq y$. We denote the power set of X by $\mathscr{P}(X)$. The element $0 \in X$ is the smallest element of X.

2.2 Properties of clones above $\mathscr{U}$ and the clone $\mathrm{Pol}(T_1)$

2.2.1 What $\langle T_1 \rangle$ is

We start by proving that the almost unary clone $\mathscr{U}$ is a so-called *binary clone*, that is, it is generated by its binary part. Thus, when investigating $[\mathscr{U}, \mathrm{Pol}(T_1)]$, we are in fact dealing with an interval of the form $[\langle \mathscr{C}^{(2)} \rangle, \mathrm{Pol}(\mathscr{C}^{(2)})]$ for $\mathscr{C}$ a clone.

Lemma 21. *The binary almost unary functions generate all almost unary functions. That is,* $\langle T_1 \rangle = \mathscr{U}$.

Proof. Trivially, $\langle T_1\rangle \subseteq \mathscr{U}$. Now we prove by induction that $\mathscr{U}^{(n)} \subseteq \langle T_1\rangle$ for all $n \geq 1$. This is obvious for $n = 1, 2$. Assume we have $\mathscr{U}^{(k)} \subseteq \langle T_1\rangle$ for all $k < n$ and take any function $f \in \mathscr{U}^{(n)}$. Say without loss of generality that $f(x_1, \dots, x_n) \leq \gamma(x_1)$ for some $\gamma \in \mathscr{O}^{(1)}$. We will use the function $p_\Delta \in T_1$ to code two variables into one and then use the induction hypothesis. Define

$$g_1(x_1, \dots, x_{n-2}, z) = \begin{cases} f(x_1, \dots, x_{n-2}, (p_\Delta^{-1}(z))_1^2, (p_\Delta^{-1}(z))_2^2) & , z \in p_\Delta[X^2] \setminus \{0\} \\ x_1 & , \text{otherwise} \end{cases}$$

The function is an element of $\mathscr{U}^{(n-1)}$ as it is bounded by $\max(x_1, \gamma(x_1))$. Intuitively, g_1 does the following: If $z \neq 0$ and in the range of p_Δ, then g_1 imagines a pair (x_{n-1}, x_n) to be coded into z via p_Δ. It reconstructs the pair (x_{n-1}, x_n) and calculates $f(x_1, \dots, x_n)$. If $z = 0$ or not in the range of p_Δ, then g knows there is no information in z; it simply forgets about the tuple $(x_2, \dots, x_n)$ and returns x_1, relying on the following similar function to do the job: Set $\Delta' = \Delta \cup \{(x, x) : x \in X\}$ and define

$$g_2(x_1, \dots, x_{n-2}, z) = \begin{cases} f(x_1, \dots, x_{n-2}, (p_{\Delta'}^{-1}(z))_2^2, (p_{\Delta'}^{-1}(z))_1^2) & , z \in p_{\Delta'}[X^2] \setminus \{0\} \\ x_1 & , \text{otherwise} \end{cases}$$

The function g_2 does exactly the same as g_1 but assumes the pair (x_{n-1}, x_n) to be coded into z in wrong order, namely as (x_n, x_{n-1}), plus it cares for the diagonal. Now consider

$$h(x_1, \dots, x_n) = g_2(g_1(x_1, \dots, x_{n-2}, p_\Delta(x_{n-1}, x_n)), x_2, \dots, x_{n-2}, p_{\Delta'}(x_n, x_{n-1})).$$

All functions which occur in h are almost unary with at most $n - 1$ variables. We claim that $h = f$. Indeed, if $x_{n-1} < x_n$, then $p_\Delta(x_{n-1}, x_n) \neq 0$ and g_1 yields f. But $p_{\Delta'}(x_n, x_{n-1}) = 0$ and so g_2 returns $g_1 = f$. If on the other hand $x_n \leq x_{n-1}$, then $p_\Delta(x_{n-1}, x_n) = 0$ and $g_1 = x_1$, whereas $p_{\Delta'}(x_n, x_{n-1}) \neq 0$, which implies $h = f(g_1, x_2, \dots, x_n) = f(x_1, \dots, x_n)$. □

The following lemma will be crucial for our investigation of clones containing T_1.

Corollary 22. *Let $\mathscr{C}$ be a clone containing T_1. Then $\mathscr{C}$ is downward closed, that is, if $f \in \mathscr{C}$, then also $g \in \mathscr{C}$ for all $g \leq f$.*

Proof. If $f \in \mathscr{C}^{(n)}$ and $g \in \mathscr{O}^{(n)}$ with $g \leq f$ are given, define $h_g(x_1, \dots, x_{n+1}) = \min(g(x_1, \dots, x_n), x_{n+1})$. Then $h_g \leq x_{n+1}$ and consequently, $h_g \in \langle T_1\rangle \subseteq \mathscr{C}$. Now $g = h_g(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathscr{C}$. □

2.2.2 Wildness of functions

We have seen in the last section that the interval $[\mathscr{U}, \mathscr{O}]$ is about growth of functions as all clones in that interval are downward closed. But mind we are not talking about how rapidly functions are growing in the sense of polynomial growth, exponential growth and so forth since we are considering clones modulo $\mathscr{O}^{(1)}$ (and so we can make functions as steep as we like); the growth of a function will be determined by which of its variables are responsible for the function to obtain many values. The following definition is due to M. Goldstern and S. Shelah [GS02]. Recall that $N = \{1, \ldots, n\}$.

Definition 23. Let $f \in \mathscr{O}^{(n)}$. We call a set $\emptyset \neq A \subseteq N$ *f-strong* iff for all $a \in X^A$ the set $\{f(a \cup x) : x \in X^{-A}\}$ is small. A is *f-weak* iff it is not f-strong. In order to use the defined notions more freely, we define the empty set to be f-strong iff f has small range.

Thus, a set of indices of variables of f is strong iff f is bounded whenever those variables are. For example, a function is almost unary iff it has a one-element strong set. Here, we shall rather think in terms of the complements of weak sets.

Definition 24. Let $f \in \mathscr{O}^{(n)}$ and let $A \subsetneqq N$ and $a \in X^{-A}$. We say A is *(f, a)-wild* iff the set $\{f(a \cup x) : x \in X^A\}$ is large. The set A is called *f-wild* iff there exists $a \in X^{-A}$ such that A is (f, a)-wild. We say that A is *f-insane* iff A is (f, a)-wild for all $a \in X^{-A}$. The set N itself we call f-wild and f-insane iff f is unbounded.

Observe that if $A \subseteq B \subseteq N$ and A is f-wild, then B is f-wild as well. Obviously, $A \subseteq N$ is f-wild iff $-A$ is f-weak. It is useful to state the following trivial criterion for a function to be almost unary.

Lemma 25. *Let $n \geq 2$ and $f \in \mathscr{O}^{(n)}$. f is almost unary iff there exists a subset of N with $n - 1$ elements which is not f-wild.*

Proof. If f is almost unary, then there is a one-element f-strong subset of N and the complement of that set is not f-wild. If on the other hand there exists $k \in N$ such that $N \setminus \{k\}$ is not f-wild, then $\{k\}$ is f-strong and so f is almost unary. $\square$

We will require the following fact from [GS02].

Fact 26. *If $f \in \mathrm{Pol}(T_1)^{(n)}$ and $A_1, A_2 \subseteq N$ are f-wild, then $A_1 \cap A_2 \neq \emptyset$.*

We observe that the converse of this statement holds as well.

Lemma 27. *Let $f \in \mathscr{O}^{(n)}$ be any n-ary function. If all pairs of f-wild subsets of N have a nonempty intersection, then $f \in \mathrm{Pol}(T_1)$.*

Proof. Let $g_1, \ldots, g_n \in T_1$ be given and set $A_1 = \{k \in N : \exists \gamma \in \mathscr{O}^{(1)}\,(g_k(x_1, x_2) \leq \gamma(x_1))\}$ and $A_2 = -A_1$. Since $A_1 \cap A_2 = \emptyset$ either A_1 or A_2 cannot be f-wild. Thus $f(g_1, \ldots, g_n)$ is bounded by a unary function of x_2 in the first case and by a unary function of x_1 in the second case. ☐

The equivalence yields a first description of $\mathrm{Pol}(T_1)$ with an interesting consequence.

Theorem 28. *A function $f \in \mathscr{O}^{(n)}$ is an element of $\mathrm{Pol}(T_1)$ iff all pairs of f-wild subsets of N have a nonempty intersection.*

2.2.3 Descriptive set theory

We show now that for countable X, this description implies that $\mathrm{Pol}(T_1)$ is a Borel set with respect to the natural topology on $\mathscr{O}$. The reader not interested in the topic can skip this part and proceed directly to the next section.

We first explain the very basics of descriptive set theory; for more details consult [Kec95]. Let $\mathscr{T} = (T, \Upsilon)$ be a *Polish space*, that is, a complete, metrizable, separable topological space. The *Borel sets* of $\mathscr{T}$ are the smallest σ-algebra on T which contains the open sets. These sets can be ordered according to their complexity: One starts by defining $\Sigma^0_1 = \Upsilon \subseteq \mathscr{P}(T)$ to consist exactly of the open sets and Π^0_1 of the closed sets. Then one continues inductively for all $1 < \alpha < \omega_1$ by setting Π^0_α to contain precisely the complements of Σ^0_α sets, and Σ^0_α to consist of all countable unions of sets which are elements of $\bigcup_{1 \leq \delta < \alpha} \Pi^0_\delta$. The sequences $(\Sigma^0_\alpha)_{1 \leq \alpha < \omega_1}$ and $(\Pi^0_\alpha)_{1 \leq \alpha < \omega_1}$ are increasing and the union over either of the two sequences yields the Borel sets.

Equip our base set $X = \omega$ with the discrete topology. Then the product space $\mathscr{N} = \omega^\omega = \mathscr{O}^{(1)}$ is the so-called *Baire space*. It is obvious that $\mathscr{O}^{(n)} = \omega^{\omega^n}$ is homeomorphic to $\mathscr{N}$. Examples of open sets in $\mathscr{O}^{(n)}$ are the $A^y_x = \{f \in \mathscr{O}^{(n)} : f(x) = y\}$, where $x \in X^n$ and $y \in X$; in fact, these sets form a subbasis of the topology of $\mathscr{O}^{(n)}$. $\mathscr{O} = \bigcup_{n=1}^{\infty} \mathscr{O}^{(n)}$ is the sum space of ω copies of $\mathscr{N}$: The open sets in $\mathscr{O}$ are those whose intersection with each $\mathscr{O}^{(n)}$ is open in $\mathscr{O}^{(n)}$. With this topology, $\mathscr{O}$ is a Polish space, and in fact again homeomorphic to $\mathscr{N}$.

Since clones are subsets of $\mathscr{O}$, they can divided into *Borel clones* and clones which are no Borel sets. In our case, we find that $\mathrm{Pol}(T_1)$ is a very simple Borel set.

Theorem 29. *Let X be countably infinite. Then $\mathrm{Pol}(T_1)$ is a Borel set in $\mathscr{O}$.*

Proof. We have to show that $\mathrm{Pol}(T_1)^{(n)}$ is Borel in $\mathscr{O}^{(n)}$ for each $n \geq 1$. By the preceding theorem,

$$\mathrm{Pol}(T_1)^{(n)} = \{f \in \mathscr{O}^{(n)} : \forall A, B \subseteq N(A, B\ f\text{-wild} \rightarrow A \cap B \neq \emptyset)\}$$

There are no (only finite) quantifiers in this definition except for those which might occur in the predicate of wildness (observe that $\exists$-quantifiers correspond to unions and $\forall$-quantifiers to intersections). Now

$$A \subseteq N\ f\text{-wild} \leftrightarrow \exists a \in X^{-A} \forall k \in X \exists b \in X^{A}(f(a \cup b) > k)$$

For fixed $A \subseteq N$, $a \in X^{-A}$, $k \in X$, and $b \in X^{A}$, the set of all functions in $\mathscr{O}^{(n)}$ for which $(f(a \cup b) > k)$ is open. Thus, the set of all $f \in \mathscr{O}^{(n)}$ for which A is f-wild is of the form $\bigcup \bigcap \bigcup$ *open*, and hence Σ^0_3 by counting of unions and negations. Observe that all unions which occur in the definition are countable.

Since the predicate of wildness is negated in the definition of $\mathrm{Pol}(T_1)^{(n)}$, we conclude that $\mathrm{Pol}(T_1)^{(n)}$ is Π^0_3. □

It is readily verified that $\mathscr{U}$ (and hence, T_1) is a Borel set as well. This is interesting in connection with the following:

Above the Borel sets of a Polish space, one can continue the hierarchy of complexity. The next level, Σ^1_1, comprises the so-called *analytic sets*, which are the continuous images of Borel sets; the *co-analytic sets* (Π^1_1) are the complements of analytic sets. It is easy to see that the clone generated by a Borel set of functions in $\mathscr{O}$ is an analytic set. Since $\mathscr{O}^{(1)}$ and all countable sets are Borel, every set which is countably generated over $\mathscr{O}^{(1)}$ is analytic. M. Goldstern showed in [Gol0x] that T_2 and $\mathrm{Pol}(T_2)$ are relatively complicated:

Theorem 30. *Let X be countably infinite. Then T_2 and $\mathrm{Pol}(T_2)$ are co-analytic but not analytic in $\mathscr{O}$. Hence, neither of the two clones $\langle T_2 \rangle$ and $\mathrm{Pol}(T_2)$ is countably generated over $\mathscr{O}^{(1)}$.*

2.2.4 What wildness means

We wish to compare the wildness of functions. Write S_N for the set of all permutations on N.

Definition 31. For $f, g \in \mathscr{O}^{(n)}$ we say that f is *as wild as* g and write $f \sim_W g$ iff there exists a permutation $\pi \in S_N$ such that A is f-wild if and only if $\pi[A]$ is g-wild for all $A \subseteq N$. Moreover, g is *at least as wild as* f ($f \leq_W g$) iff there is a permutation $\pi \in S_N$ such that for all f-wild subsets $A \subseteq N$ the image $\pi[A]$ of A under π is g-wild.

Lemma 32. *$\sim_W$ is an equivalence relation and $\leq_W$ a quasiorder extending $\leq$ on the set of n-ary functions $\mathscr{O}^{(n)}$.*

Proof. We leave the verification of this to the reader. $\square$

Lemma 33. *Let $f, g \in \mathscr{O}^{(n)}$. Then $f \sim_W g$ iff $f \leq_W g$ and $g \leq_W f$.*

Proof. It is clear that $f \leq_W g$ (and $g \leq_W f$) if $f \sim_W g$. Now assume $f \leq_W g$ and $g \leq_W f$. Then there are $\pi_1, \pi_2 \in S_N$ which take f-wild and g-wild subsets of N to g-wild and f-wild sets, respectively.

Set $\pi = \pi_2 \circ \pi_1$. Then A is f-wild iff $\pi[A]$ is f-wild for any subset A of N: If A is f-wild, then $\pi_1[A]$ is g-wild, then $\pi_2[\pi_1[A]] = \pi[A]$ is f-wild. If on the other hand $\pi[A]$ is f-wild, then take $k \geq 1$ such that $\pi^k = id_N$ and observe that $\pi^{k-1}[\pi[A]] = \pi^k[A] = A$ is f-wild.

Now we see that A is f-wild iff $\pi_1[A]$ is g-wild for all $A \subseteq N$: If $\pi_1[A]$ is g-wild, then so is $\pi_2 \circ \pi_1[A] = \pi[A]$ and so is A by the preceding observation. Hence, the permutation π_1 shows that $f \sim_W g$. $\square$

Corollary 34. *Let $n \geq 1$. Then $\leq_W/\sim_W$ is a partial order on the $\sim_W$-equivalence classes of $\mathscr{O}^{(n)}$.*

Notation 35. Let $f \in \mathscr{O}^{(n)}$. By $\langle f \rangle_{T_1}$ we mean $\langle \{f\} \cup T_1 \rangle$ from now on. $\langle f \rangle_{T_1}$ is the smallest clone containing f as well as all almost unary functions.

We are aiming for the following theorem which tells us why we invented wildness.

Theorem 36. *Let $f, g \in \mathscr{O}^{(n)}$. If $f \leq_W g$, then $f \in \langle g \rangle_{T_1}$. In words, if g is at least as wild as f, then it generates f modulo T_1.*

Corollary 37. *Let $f, g \in \mathscr{O}^{(n)}$. If $f \sim_W g$, then $\langle f \rangle_{T_1} = \langle g \rangle_{T_1}$.*

We split the proof of Theorem 36 into a sequence of lemmas. In the next lemma we see that it does not matter which $a \in X^{-A}$ makes a set $A \subseteq N$ wild.

Lemma 38. *Let $g \in \mathscr{O}^{(n)}$. Then there exists $g' \in \langle g \rangle_{T_1}^{(n)}$ such that for all $A \subseteq N$ the following holds: If A is g-wild, then A is $(g', 0^{-A})$-wild.*

Proof. Fix for all g-wild $A \subseteq N$ a tuple $a_A \in X^{-A}$ such that $\{g(x \cup a_A) : x \in X^A\}$ is large. For an n-tuple $(x_1, \ldots, x_n)$ write $P = P(x_1, \ldots, x_n) = \{l \in N : x_l \neq 0\}$ for the set of indices of positive components in the tuple. Define for $1 \leq i \leq n$ functions

$$\gamma_i(x_1, \ldots, x_n) = \begin{cases} x_i & , x_i \neq 0 \vee P(x_1, \ldots, x_n) \text{ not } g\text{-wild} \\ (a_P)_i & , \text{otherwise} \end{cases}$$

In words, if the set P of indices of positive components in $(x_1, \dots, x_n)$ is a wild set, then the γ_i leave those positive components alone and send the zero components to the respective values making P wild. Otherwise, they act just like projections. It is obvious that γ_i is almost unary, $1 \le i \le n$. Set $g' = g(\gamma_1, \dots, \gamma_n) \in \langle g \rangle_{T_1}$. To prove that g' has the desired property, let $A \subseteq N$ be g-wild. Choose any minimal g-wild $A' \subseteq A$. Then by the definition of wildness the set $\{g(x \cup a_{A'}) : x \in X^{A'}\}$ is large. Take a large $B \subseteq X^{A'}$ such that the sequence $(g(x \cup a_{A'}) : x \in B)$ is one-one. Select further a large $C \subseteq B$ such that each component in the sequence of tuples $(x : x \in C)$ is either constant or injective and such that 0 does not occur in any of the injective components (it is a simple combinatorial fact that this is possible). If one of the components were constant, then A' would not be minimal g-wild; hence, all components are injective. Now we have

$$\begin{aligned} |X| &= |\{g(x \cup a_{A'}) : x \in C\}| \\ &= |\{g'(x \cup 0^{-A'}) : x \in C\}| \quad \le |\{g'(x \cup 0^{-A}) : x \in X^A\}| \end{aligned}$$

and so A is $(g', 0^{-A})$-wild. □

We prove that we can assume functions to be monotone.

Lemma 39. *Let $g \in \mathscr{O}^{(n)}$. Then there exists $g'' \in \langle g \rangle_{T_1}^{(n)}$ such that $g \le g''$ and g'' is monotone with respect to the pointwise order $\le$.*

Proof. We will define a mapping γ from X^n to X^n such that $\gamma_i = \pi_i^n \circ \gamma$ is almost unary for $1 \le i \le n$ and such that $g'' = g \circ \gamma$ has the desired property. We fix for every g-wild $A \subseteq N$ a sequence $(\alpha_\xi^A)_{\xi \in X}$ of elements of X^n so that all components of α_ξ^A which lie not in A are constant and so that $(g(\alpha_\xi^A))_{\xi \in X}$ is monotone and unbounded.

Let $x \in X^n$. The *order type* of x is the unique n-tuple $(j_1, \dots, j_n)$ of indices in N such that $\{j_1, \dots, j_n\} = \{1, \dots, n\}$ and such that $x_{j_1} \le \dots \le x_{j_n}$ and such that $j_k < j_{k+1}$ whenever $x_{j_k} = x_{j_{k+1}}$. Let $1 \le k \le n$ be the largest element with the property that the set $\{j_k, \dots, j_n\}$ is g-wild. We call the set $\{j_k, \dots, j_n\}$ the *pushing set* $Push(x)$ and $\{j_1, \dots, j_{k-1}\}$ the *holding set* of x with respect to g.

We define by transfinite recursion

$$\gamma : \begin{array}{ccc} X^n & \to & X^n \\ x & \mapsto & \alpha_{\lambda(x)}^{Push(x)} \end{array}$$

where

$$\lambda(x) = \min\{\xi : g(\alpha_\xi^{Push(x)}) \ge \sup(\{g''(y) : y < x\} \cup \{g(x)\})\}.$$

This looks worse than it is: We simply map x to the first element of the sequence $(\alpha_\xi^{Push(x)})_{\xi\in X}$ such that all values of g'' already defined as well as $g(x)$ are topped. By definition, $g'' = g \circ \gamma$ is monotone and $g \leq g''$. It only remains to prove that all γ_i, $1 \leq i \leq n$, are almost unary to see that $g'' \in \langle g\rangle_{T_1}$.

Suppose not, and say that γ_k is not almost unary for some $1 \leq k \leq n$. Then there exists a value $c \in X$ and a sequence of n-tuples $(\beta_\xi)_{\xi\in X}$ with constant value c in the k-th component such that $(\gamma_k(\beta_\xi))_{\xi\in X}$ is unbounded. Since there exist only finitely many order types of n-tuples, we can assume that all β_ξ have the same order type $(j_1,\ldots,j_n)$; say without loss of generality $(j_1,\ldots,j_n) = (1,\ldots,n)$. Then all β_ξ have the same pushing set $Push(\beta)$ of indices. If k was an element of the holding set of the tuples β_ξ, then $(\gamma_k(\beta_\xi) : \xi \in X)$ would be constant so that k must be in $Push(\beta)$. Clearly, $(\lambda(\beta_\xi))_{\xi\in X}$ has to be unbounded as otherwise $(\gamma_k(\beta_\xi))_{\xi\in X}$ would be bounded. Since by definition the value of λ increases only when it is necessary to keep $g \leq g''$, the set $\{g(y) : \exists\xi \in X(y \leq \beta_\xi)\}$ is unbounded. But because of the order type of the β_ξ, whenever $i \leq k$, then we have $(\beta_\xi)_i^n \leq c$ for all $\xi \in X$ so that the components of the β_ξ with index in the set $\{1,\ldots,k\}$ are bounded. Thus, $\{k+1,\ldots,n\}$ is g-wild, contradicting the fact that k is in the pushing set $Push(\beta)$.

□

In a next step we shall see that modulo T_1, wildness is insanity.

Lemma 40. *Let $g \in \mathscr{O}^{(n)}$. Then there exists $g'' \in \langle g\rangle_{T_1}^{(n)}$ such that g'' is monotone and for all $A \subseteq N$ the following holds: If A is g-wild, then A is g''-insane.*

Proof. Let $g' \in \langle g\rangle_{T_1}^{(n)}$ be provided by Lemma 38 and make a monotone g'' out of it with the help of the preceding lemma. We claim that g'' already has both desired properties. To prove this, consider an arbitrary g-wild $A \subseteq N$. By construction of g', A is $(g', 0^{-A})$-wild and so it is also $(g'', 0^{-A})$-wild as $g' \leq g''$. But $0^{-A} \leq a$ for all $a \in X^{-A}$; hence the fact that g'' is monotone implies that A is (g, a)-wild for all $a \in X^{-A}$ which means exactly that A is g''-insane. □

Lemma 41. *Let $f, g \in \mathscr{O}^{(n)}$. If $f \leq_W g$, then there exists $h \in \langle g\rangle_{T_1}^{(n)}$ such that $f \leq h$.*

Proof. Without loss of generality, we assume that the permutation $\pi \in S_N$ taking f-wild subsets of N to g-wild sets is the identity on N. We take $g'' \in \langle g\rangle_{T_1}$ according to the preceding lemma. We wish to define $\gamma \in \mathscr{O}^{(1)}$ with $f \leq \gamma \circ g''$. For $x \in X$ write $U_x = g''^{-1}[\{x\}]$ for the preimage of x under g''. Now set

$$\gamma(x) = \begin{cases} \sup\{f(y) : y \in U_x\} & , U_x \neq \emptyset \\ 0 & , \text{otherwise} \end{cases}$$

We claim that γ is well-defined, that is, the supremum in its definition always exists in X. For suppose there is an $x \in X$ such that the set $\{f(y) : y \in U_x\}$ is unbounded. Choose a large subset $B \subseteq U_x$ making the sequence $(f(y) : y \in B)$ one-one. Take further a large $C \subseteq B$ so that all components in the sequence $(y : y \in C)$ are either one-one or constant. Set $A \subseteq N$ to consist of the indices of the injective components. Obviously, A is f-wild; therefore it is g''-insane. Since g'' is also monotone, the set $\{g''(y) : y \in C\}$ is large, contradicting the fact that g'' is constant on U_x. Thus, γ is well-defined and clearly $f \leq h \in \langle g \rangle_{T_1}$ where $h = \gamma \circ g''$. □

Proof of Theorem 36. The assertion is an immediate consequence of the preceding lemma and the fact that all clones above $\mathscr{U}$ are downward closed. □

Remark 42. Unfortunately, the converse does not hold: If $f, g \in \mathscr{O}^{(n)}$ and $f \in \langle g \rangle_{T_1}$ then it need not be true that $f \leq_W g$. We will see an example at the end of the section.

2.2.5 med_3 and T_1 generate $\mathrm{Pol}(T_1)$

We are now ready to prove the explicit description of $\mathrm{Pol}(T_1)$.

Definition 43. For all $n \geq 1$ and all $1 \leq k \leq n$ we define a function

$$m_k^n(x_1, \ldots, x_n) = x_{j_k} \quad , \text{if } x_{j_1} \leq \ldots \leq x_{j_n}.$$

For example, m_n^n is the maximum function $\max_n$ and m_1^n the minimum function $\min_n$ in n variables. Note that $\min_n \in \mathrm{Pol}(T_1)$ (it is even almost unary) but $\max_n \notin \mathrm{Pol}(T_1)$ (and hence $\langle \max_n \rangle_{T_1} = \mathscr{O}$). If n is an odd number then we call $m^n_{\frac{n+1}{2}}$ the n-th median function and denote this function by med_n.

For fixed odd n it is easily verified (check the wild sets and apply Theorem 28) that med_n it is the largest of the m_k^n which still lies in $\mathrm{Pol}(T_1)$: $m_k^n \in \mathrm{Pol}(T_1)$ iff $k \leq \frac{n+1}{2}$. It is for this reason that we are interested in the median functions on our quest for a nice generating system of $\mathrm{Pol}(T_1)$. As a consequence of the following theorem from the preceding chapter (Theorem 2) it does not matter which of the median functions we consider:

Theorem 44. *Let $k, n \geq 3$ be odd natural numbers. Then $\mathrm{med}_k \in \langle \{\mathrm{med}_n\} \rangle$. In other words, a clone contains either no median function or all median functions.*

The following lemma states that within the restrictions of functions of $\mathrm{Pol}(T_1)$ (Fact 26), we can construct functions of arbitrary wildness with the median.

Lemma 45. *Let $n \geq 1$ and let $\mathscr{A} = \{A_1, \ldots, A_k\} \subseteq \mathscr{P}(N)$ be a set of subsets of N with the property that $A_i \cap A_j \neq \emptyset$ for all $1 \leq i, j \leq k$. Then there exists monotone $t_{\mathscr{A}} \in \langle\{\mathrm{med}_3\}\rangle^{(n)}$ such that all members of $\mathscr{A}$ are $t_{\mathscr{A}}$-insane.*

Proof. We prove this by induction over the size k of $\mathscr{A}$. If $\mathscr{A}$ is empty there is nothing to show. If $k = 1$, we can set $t_{\mathscr{A}} = \pi_i^n$, where i is an arbitrary element of A_1. Then A_1 is obviously $t_{\mathscr{A}}$-insane. If $k = 2$, then define $t_{\mathscr{A}} = \pi_i^n$, where $i \in A_1 \cap A_2$ is arbitrary. Clearly, both A_1 and A_2 are $t_{\mathscr{A}}$-insane. Finally, assume $k \geq 3$. By induction hypothesis, there exist monotone terms $t_{\mathscr{B}}, t_{\mathscr{C}}, t_{\mathscr{D}} \in \langle\{\mathrm{med}_3\}\rangle^{(n)}$ for the sets $\mathscr{B} = \{A_1, \ldots, A_{k-1}\}$, $\mathscr{C} = \{A_1, \ldots, A_{k-2}, A_k\}$ and $\mathscr{D} = \{A_{k-1}, A_k\}$ such that all sets in $\mathscr{B}$ (and $\mathscr{C}, \mathscr{D}$ respectively) are $t_{\mathscr{B}}$-insane ($t_{\mathscr{C}}$-insane, $t_{\mathscr{D}}$-insane). Set

$$t_{\mathscr{A}} = \mathrm{med}_3(t_{\mathscr{B}}, t_{\mathscr{C}}, t_{\mathscr{D}}).$$

Then each A_i is insane for two of the three terms in med_3. Thus, if we fix the variables outside A_i to arbitrary values, then at least two of the three subterms in med_3 are still unbounded and so is $t_{\mathscr{A}}$ by the monotonicity of its subterms. Hence, every A_i is $t_{\mathscr{A}}$-insane, $1 \leq i \leq k$. Obviously $t_{\mathscr{A}}$ is monotone. □

Lemma 46. *Let $f \in \mathrm{Pol}(T_1)^{(n)}$. Then there exists $t_f \in \langle\{\mathrm{med}_3\}\rangle$ such that $f \leq_W t_f$.*

Proof. Write $\mathscr{A} = \{A_1, \ldots, A_k\}$ for the set of f-wild subsets of N. By Fact 26, $A_i \cap A_j \neq \emptyset$ for all $1 \leq i, j \leq k$. Apply the preceding lemma to $\mathscr{A}$. □

Theorem 47. $\mathrm{Pol}(T_1) = \langle \mathrm{med}_3 \rangle_{T_1}$.

Proof. It is clear that $\mathrm{Pol}(T_1) \supseteq \langle \mathrm{med}_3 \rangle_{T_1}$. On the other hand we have just seen that if $f \in \mathrm{Pol}(T_1)$, then there exists $t_f \in \langle\{\mathrm{med}_3\}\rangle$ such that $f \leq_W t_f$, whence $f \in \langle \mathrm{med}_3 \rangle_{T_1}$. □

Corollary 48. $\mathrm{Pol}(T_1)$ *is the $\leq$-downward closure of the clone generated by* med_3 *and the unary functions $\mathscr{O}^{(1)}$.*

Proof. Given $f \in \mathrm{Pol}(T_1)$, by Lemma 46 there exists $t_f \in \langle\{\mathrm{med}_3\}\rangle$ such that $f \leq_W t_f$. By Lemma 45, t_f is monotone and each t_f-wild set is in fact even t_f-insane. Now one follows the proof of Lemma 41 to obtain $\gamma \in \mathscr{O}^{(1)}$ such that $f \leq \gamma \circ t_f$. □

Corollary 49. $\mathrm{Pol}(T_1) = \langle\{\mathrm{med}_3, p_\Delta\} \cup \mathscr{O}^{(1)}\rangle$. *In particular,* $\mathrm{Pol}(T_1)$ *is finitely generated over the unary functions.*

Proof. Remember that $\langle\{p_\Delta\} \cup \mathscr{O}^{(1)}\rangle = \langle T_1 \rangle$ (Fact 20) and apply Theorem 47. □

Now we can give the example promised in Remark 42. Set

$$g(x_1, \dots, x_4) = \mathrm{med}_3(x_1, x_2, x_3)$$

and

$$f(x_1, \dots, x_4) = \mathrm{med}_5(x_1, x_1, x_2, x_3, x_4).$$

It is obvious that $\langle g \rangle_{T_1} = \langle \mathrm{med}_3 \rangle_{T_1} = \mathrm{Pol}(T_1)$. Next observe that $\langle f \rangle_{T_1} \subseteq \langle \mathrm{med}_5 \rangle_{T_1} = \mathrm{Pol}(T_1)$ and that $f(x_1, x_2, x_3, x_3) = \mathrm{med}_3$ which implies $\mathrm{Pol}(T_1) = \langle \mathrm{med}_3 \rangle_{T_1} \subseteq \langle f \rangle_{T_1}$. Thus, $\langle g \rangle_{T_1} = \langle f \rangle_{T_1}$. Consider on the other hand the 2-element wild sets of the two functions: Exactly $\{1, 2\}, \{1, 3\}$ and $\{2,3\}$ are g-wild, and $\{1, 2\}, \{1, 3\}, \{1, 4\}$ are the wild sets of two elements for f. Now the intersection of first group is empty, whereas the one of the second group is not; so there is no permutation of the set $\{1, 2, 3, 4\}$ which takes the first group to the second or the other way. Hence, neither $f \leq_W g$ nor $g \leq_W f$.

2.3 The interval $[\mathscr{U}, \mathscr{O}]$

2.3.1 A chain in the interval

Now we shall show that the open interval $(\langle T_1 \rangle, \mathrm{Pol}(T_1))$ is not empty by exhibiting a countably infinite descending chain therein with intersection $\mathscr{U}$.

Notation 50. For a natural number $n \geq 2$, we write $\mathscr{M}_n = \langle \{m_2^n\} \cup T_1 \rangle$.

Observe that since $m_2^2 = \max_2 \notin \mathrm{Pol}(T_1)$, Theorem 17 implies that $\mathscr{M}_2 = \mathscr{O}$. Moreover, $m_2^3 = \mathrm{med}_3$ and hence, $\mathscr{M}_3 = \mathrm{Pol}(T_1)$.

Lemma 51. *Let $n \geq 2$. Then $\mathscr{M}_n^{(k)} = \mathscr{U}^{(k)}$ for all $1 \leq k < n$. That is, all functions in $\mathscr{M}_n$ of arity less than n are almost unary.*

Proof. Given n, k we show by induction over terms that if $t \in \mathscr{M}_n^{(k)}$, then t is almost unary. To start the induction we note that the only k-ary functions in the generating set of $\mathscr{M}_n$ are almost unary. Now assume $t = f(t_1, t_2)$, where $f \in T_1$ and $t_1, t_2 \in \mathscr{M}_n^{(k)}$. By induction hypothesis, t_1 and t_2 are almost unary and so is t as the almost unary functions are closed under composition. Finally, say $t = m_2^n(t_1, \dots, t_n)$, where the t_i are almost unary k-ary functions, $1 \leq i \leq n$. Since $k < n$, there exist $i, j \in N$ with $i \neq j$, $l \in \{1, \dots, k\}$ and $\gamma, \delta \in \mathscr{O}^{(1)}$ such that $t_i \leq \gamma(x_l)$ and $t_j \leq \delta(x_l)$. Then, $t \leq \max(\gamma, \delta)(x_l)$ and so t is almost unary as well. ☐

Corollary 52. *If $n \geq 2$, then $m_2^n \notin \mathscr{M}_{n+1}$. Consequently, $\mathscr{M}_n \nsubseteq \mathscr{M}_{n+1}$.*

Lemma 53. *If $n \geq 2$, then $m_2^{n+1} \in \mathscr{M}_n$. Consequently, $\mathscr{M}_{n+1} \subseteq \mathscr{M}_n$.*

Proof. Set

$$f(x_1, \ldots, x_{n+1}) = m_2^n(x_1, \ldots, x_n) \in \mathscr{M}_n.$$

Then every n-element subset of $\{1, \ldots, n+1\}$ is f-wild. Hence, $m_2^{n+1} \leq_W f$ and so $m_2^{n+1} \in \langle f \rangle_{T_1} \subseteq \mathscr{M}_n$. □

Theorem 54. *The sequence $(\mathscr{M}_n)_{n \geq 2}$ forms a countably infinite descending chain:*

$$\mathscr{O} = \mathscr{M}_2 \supsetneqq \mathscr{M}_3 = \mathrm{Pol}(T_1) \supsetneqq \mathscr{M}_4 \supsetneqq \ldots \supsetneqq \mathscr{M}_n \supsetneqq \mathscr{M}_{n+1} \supsetneqq \ldots$$

Moreover,

$$\bigcap_{n \geq 2} \mathscr{M}_n = \mathscr{U}.$$

Proof. The first statement follows from Corollary 52 and Lemma 53. The second statement a direct consequence of Lemma 51. □

2.3.2 Finally, this is the interval

We will now prove that there are no more clones in the interval $[\mathscr{U}, \mathscr{O}]$ than the ones we already exhibited. We first state a technical lemma.

Lemma 55. *Let $f \in \mathscr{O}^{(n)}$ be a monotone function such that all f-wild subsets of N are f-insane. Define for $i, j \in N$ with $i \neq j$ functions*

$$f^{(i,j)}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n)$$

which replace the i-th by the j-th component and calculate f. Then the following implications hold for all f-wild $A \subseteq N$ and all $i, j \in N$ with $i \neq j$:

(i) If $i \notin A$, then A is $f^{(i,j)}$-insane.

(ii) If $j \in A$, then A is $f^{(i,j)}$-insane.

Proof. We have to show that if we fix the variables outside A to constant values, then $f^{(i,j)}$ is still unbounded; because f is monotone, we can assume all values are fixed to 0. Fix a sequence $(\alpha_\xi : \xi \in X)$ of elements of X^n such that all components outside A are zero for all tuples of the sequence and such that $(f(\alpha_\xi) : \xi \in X)$ is unbounded. Define a sequence of n-tuples $(\beta_\xi : \xi \in X)$ by

$$(\beta_\xi)_k^n = \begin{cases} 0 & , k \notin A \\ \xi & , \text{otherwise} \end{cases}$$

For each $\xi \in X$ there exist a $\lambda \in X$ such that $\alpha_\xi \leq \beta_\lambda$. Then $f(\alpha_\xi) \leq f(\beta_\lambda)$. In either of the cases (i) or (ii), $f(\beta_\lambda) \leq f^{(i,j)}(\beta_\lambda)$. Thus, $(f^{(i,j)}(\beta_\xi) : \xi \in X)$ is unbounded. $\square$

Lemma 56. *Let* $f \in \mathscr{O}^{(n)}$ *not almost unary. Then there exists* $n_0 \geq 2$ *such that* $\langle f \rangle_{T_1} = \langle m_2^{n_0} \rangle_{T_1}$.

Proof. We shall prove this by induction over the arity n of f. If $n = 1$, there are no not almost unary functions so there is nothing to show. Now assume our assertion holds for all $1 \leq k < n$. We distinguish two cases:

First, consider f such that all f-wild subsets of N have size at least $n-1$. Then $f \sim_W m_2^n$ and so $\langle f \rangle_{T_1} = \langle m_2^n \rangle_{T_1}$.

Now assume there exists an f-wild subset of N of size $n-2$, say without loss of generality that $\{2, \ldots, n-1\}$ is such a set. By Lemma 40 and Theorem 36 there exists a monotone $\hat{f}$ with $\langle f \rangle_{T_1} = \langle \hat{f} \rangle_{T_1}$ and with the property that all f-wild subsets of N are $\hat{f}$-insane. Since we could replace f by $\hat{f}$, we assume that f is monotone and that all f-wild sets are f-insane.

Consider the $f^{(i,j)}$ as defined in the preceding lemma. Formally, these functions are still n-ary, but in fact they depend only on $n-1$ variables. Thus, all of the $f^{(i,j)}$ which are not almost unary satisfy the induction hypothesis. Set

$$n_0 = \min\{k : \exists i, j \in N \ \langle f^{(i,j)} \rangle_{T_1} = \langle m_2^k \rangle_{T_1}\}.$$

The minimum is well-defined: Because $\{2, \ldots, n-1\}$ is f-insane, $f^{(n,1)}$ is not almost unary so that it generates the same clone as some m_2^n modulo T_1; thus, the set is not empty. Clearly, $m_2^{n_0} \in \langle f \rangle_{T_1}$. We show that $m_2^{n_0}$ is strong enough to generate f. Since $\mathscr{M}_n \subseteq \mathscr{M}_{n_0}$ for all $n \geq n_0$ we have $f^{(i,j)} \in \langle m_2^{n_0} \rangle_{T_1}$ for all $i, j \in N$ with $i \neq j$. Now define

$$t(x_1, \ldots, x_n) = f^{(n,1)}(x_1, f^{(1,2)}, f^{(1,3)}, \ldots, f^{(1,n-1)}) \in \langle m_2^{n_0} \rangle_{T_1}.$$

We claim that $f \leq_W t$. Indeed, let $A \subseteq N$ be f-wild and whence f-insane by our assumption.

If $1 \notin A$, then A is $f^{(1,j)}$-insane for all $2 \leq j \leq n-1$ by the preceding lemma. So A is insane for all components in the definition of t except the first one. Hence, because f is monotone, A must be t-insane as otherwise $f^{(n,1)}$ would be almost unary.

If $1 \in A$, then by the preceding lemma A is still $f^{(1,j)}$-insane whenever $j \in A$. Thus, increasing the components with index in A increases the first component in t plus all subterms $f^{(1,j)}$ with $j \in A$; but by the definition of $f^{(n,1)}$, that is the same as increasing the variables $A \cup \{n\} \supseteq A$ in f. Whence, A is t-insane.

This proves $f \leq_W t$ and thus $f \in \langle m_2^{n_0} \rangle_{T_1}$. $\square$

So here it is, the interval and the end of our quest.

Theorem 57. *Let $\mathscr{C} \supsetneqq \mathscr{U}$ be a clone. Then there exists $n \geq 2$ such that $\mathscr{C} = \mathscr{M}_n$.*

Proof. Set

$$n_{\mathscr{C}} = \min\{n \geq 2 : \mathscr{M}_n \subseteq \mathscr{C}\}.$$

Since $\mathscr{C}$ contains a function which is not almost unary, the preceding lemma implies that the set over which we take the minimum is nonempty. Obviously, $\mathscr{M}_{n_{\mathscr{C}}} \subseteq \mathscr{C}$. Now let f be an arbitrary function in $\mathscr{C}$ which is not almost unary. Then by the preceding lemma, there exists n_0 such that $\langle m_2^{n_0} \rangle_{T_1} = \langle f \rangle_{T_1}$. Clearly, $n_0 \geq n_{\mathscr{C}}$ so that $f \in \mathscr{M}_{n_0} \subseteq \mathscr{M}_{n_{\mathscr{C}}}$. ☐

We state a lemma describing how the k-ary parts of the $\mathscr{M}_n$ for arbitrary k relate to each other.

Lemma 58. *Let $m > n \geq 2$ and $k \geq 2$. If $k \geq n$ (that is, if $\mathscr{M}_n^{(k)}$ is nontrivial), then $\mathscr{M}_n^{(k)} \supsetneqq \mathscr{M}_m^{(k)}$.*

Proof. We know that $\mathscr{M}_n^{(k)} \supseteq \mathscr{M}_m^{(k)}$. To see the inequality of the two sets, observe that

$$f(x_1, \ldots, x_k) = m_2^n(x_1, \ldots, x_n)$$

is an element of $\mathscr{M}_n^{(k)}$ but definitely not one of $\mathscr{M}_m^{(k)}$. ☐

Corollary 59. *Let $k \geq 2$. Then*

$$\mathscr{M}_2^{(k)} \supsetneqq \mathscr{M}_3^{(k)} \supsetneqq \ldots \supsetneqq \mathscr{M}_k^{(k)} \supsetneqq \mathscr{M}_{k+1}^{(k)} = \mathscr{U}^{(k)}$$

Consequently, there are k different k-ary parts of clones of the interval $[\mathscr{U}, \mathscr{O}]$ for each k.

In general, if $\mathscr{C}$ is a clone, then

$$\mathrm{Pol}(\mathscr{C}^{(1)}) \supseteq \mathrm{Pol}(\mathscr{C}^{(2)}) \supseteq \ldots \supseteq \mathrm{Pol}(\mathscr{C}^{(n)}) \supseteq \ldots$$

Moreover,

$$\mathrm{Pol}(\mathscr{C}^{(n)})^{(n)} = \mathscr{C}^{(n)} \quad \text{and} \quad \bigcap_{n \geq 1} \mathrm{Pol}(\mathscr{C}^{(n)}) = \mathscr{C}.$$

It is natural to ask whether or not for $\mathscr{C} = \mathscr{U}$ this chain coincides with the chain we discovered.

Theorem 60. *Let $n \geq 1$. Then $\mathscr{M}_{n+1} = \mathrm{Pol}(\mathscr{U}^{(n)})$.*

Proof. Clearly, $\mathscr{M}_2 = \mathrm{Pol}(\mathscr{U}^{(1)}) = \mathscr{O}$, so assume $n \geq 2$. Consider m_2^{n+1} and let $f_1, \ldots, f_{n+1}$ be functions in $\mathscr{U}^{(n)}$. Then two of the f_j are bounded by unary functions of the same variable. Thus $m_2^{n+1}(f_1, \ldots, f_{n+1})$ is bounded by a unary function of this variable. This shows $m_2^{n+1} \in \mathrm{Pol}(\mathscr{U}^{(n)})$ and hence $\mathscr{M}_{n+1} \subseteq \mathrm{Pol}(\mathscr{U}^{(n)})$. Now consider m_2^n and observe that $m_2^n \notin \mathscr{U}^{(n)} = \mathrm{Pol}(\mathscr{U}^{(n)})^{(n)}$; this proves $\mathscr{M}_n \nsubseteq \mathrm{Pol}(\mathscr{U}^{(n)})$. Whence, $\mathscr{M}_{n+1} = \mathrm{Pol}(\mathscr{U}^{(n)})$. □

2.3.3 The m_k^n in the chain

As an example, we will show where the clones generated by the m_k^n (as in Definition 43) and T_1 can be found in the chain.

Notation 61. For $1 \leq k \leq n$ we set $\mathscr{M}_n^k = \langle m_k^n \rangle_{T_1}$.

Note that if $k = 1$, then $\mathscr{M}_n^k = \mathscr{U}$, and if $k > \frac{n+1}{2}$, then $\mathscr{M}_n^k = \mathscr{O}$. Observe also that $\mathscr{M}_n = \mathscr{M}_n^2$ for all $n \geq 2$.

Notation 62. For a positive rational number q we write

$$\lfloor q \rfloor = \max\{n \in \mathbb{N} : \, n \leq q\}$$

and

$$\lceil q \rceil = \min\{n \in \mathbb{N} : \, q \leq n\}.$$

The remainder of the division $\frac{n}{k}$ we denote by the symbol $R(\frac{n}{k})$.

Lemma 63. *Let $2 \leq k \leq \frac{n+1}{2}$ and let $t \in \mathscr{M}_n^k$ not almost unary. Then all t-wild subsets of N_t have size at least $\frac{n}{k-1} - 1$.*

Proof. Our proof will be by induction over terms. If $t = m_k^n$, then all t-wild subsets of $N_t = N$ have at least $n - k + 1$ elements in accordance with our assertion. For the induction step, assume $t = f(t_1, t_2)$, where $f \in T_1$, say $f(x_1, x_2) \leq \gamma(x_1)$ for some $\gamma \in \mathscr{O}^{(1)}$. Then t inherits the asserted property from t_1. Finally we consider the case where $t = m_k^n(t_1, \ldots, t_n)$. Suppose towards contradiction there exists $A \subseteq N_t$ t-wild with $|A| < \frac{n}{k-1} - 1$. There have to be at least $n - k + 1$ terms t_j for which A is t_j-wild so that A can be t-wild. By induction hypothesis, these $n - k + 1$ terms are almost unary and bounded by a unary function of a variable with index in A. From the bound on the size of A we conclude that at least

$$\lceil \frac{n-k+1}{|A|} \rceil > \frac{n-k+1}{\frac{n}{k-1} - 1} = k - 1$$

of the terms t_j are bounded by an unary function of the same variable with index in A. But if k of the t_j have the same one-element strong set, then t is bounded by a unary function of this variable as well in contradiction to the assumption that t is not almost unary. $\square$

Corollary 64. *Let* $2 \leq k \leq \frac{n+1}{2}$*. Then* $\mathscr{M}_{\lceil \frac{n}{k-1} \rceil - 1} \nsubseteq \mathscr{M}_n^k$*.*

Proof. With the preceding lemma it is enough to observe that $m_2^{\lceil \frac{n}{k-1} \rceil - 1} \in \mathscr{M}_{\lceil \frac{n}{k-1} \rceil - 1}$ has a wild set of size $\lceil \frac{n}{k-1} \rceil - 2$. $\square$

So we identify now the $\mathscr{M}_j$ which $\mathscr{M}_n^k$ is equal to.

Lemma 65. *Let* $2 \leq k \leq n$*. Then* $\mathscr{M}_{\lceil \frac{n}{k-1} \rceil} \subseteq \mathscr{M}_n^k$*.*

Proof. It suffices to show that m_k^n generates $m_2^{\lceil \frac{n}{k-1} \rceil}$. But this is easy:

$$m_2^{\lceil \frac{n}{k-1} \rceil} = m_k^n(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_{\lceil \frac{n}{k-1} \rceil}, \ldots, x_{\lceil \frac{n}{k-1} \rceil}),$$

where x_j occurs $k-1$ times if $1 \leq j \leq \lfloor \frac{n}{k-1} \rfloor$ and $R(\frac{n}{k-1}) < k-1$ times if $j = \lfloor \frac{n}{k-1} \rfloor + 1$. For if we evaluate the function for a $\lceil \frac{n}{k-1} \rceil$-tuple with $x_{j_1} \leq \ldots \leq x_{j_{\lceil \frac{n}{k-1} \rceil}}$, then x_{j_1} occurs at most $k-1$ times in the tuple, but x_{j_1} together with x_{j_2} occur more than k times; thus, the k-th smallest element in the tuple is x_{j_2} and m_k^n returns x_{j_2}. $\square$

Theorem 66. $\mathscr{M}_n^k = \mathscr{M}_{\lceil \frac{n}{k-1} \rceil}$ *for all* $2 \leq k \leq n$*.*

Proof. By Theorem 57, $\mathscr{M}_n^k$ has to be somewhere in the chain $(\mathscr{M}_n)_{n \geq 2}$. Because of Corollary 64 and Lemma 65 the assertion follows. $\square$

2.3.4 Further on the chain

We conclude by giving one simple guideline for where to search the clone $\langle f \rangle_{T_1}$ in the chain for arbitrary $f \in \mathscr{O}$.

Lemma 67. *Let* $1 \leq k \leq n$ *and let* $f \in \mathscr{O}^{(n)}$ *be a not almost unary function which has a* k*-element* f*-wild subset of* N*. Then* $\mathscr{M}_{k+1} \subseteq \langle f \rangle_{T_1}$*.*

Proof. We can assume that $\{1, \ldots, k\}$ and all $A \subseteq N$ with $|A| = n-1$ are f-insane and that f is monotone. Define

$$g(x_1, \ldots, x_{k+1}) = f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+1}) \in \langle f \rangle_{T_1}.$$

Let $A \subseteq \{1,\dots,k+1\}$ with $|A| = k$ be given. If $A = \{1,\dots,k\}$ then A is f-wild and so it is g-wild. Otherwise A contains $k+1$ and so it affects $n-1$ components in the definition of g. Therefore A is g-wild by Lemma 25. Hence, $m_2^{k+1} \leq_W g$ and so $\mathscr{M}_{k+1} \subseteq \langle g \rangle_{T_1} \subseteq \langle f \rangle_{T_1}$. □

Remark 68. Certainly it is not true that if the smallest wild set of a function $f \in \mathscr{O}$ has k elements, then $\mathscr{M}_{k+1} = \langle f \rangle_{T_1}$. The m_k^n are an example.

Corollary 69. *Let $f \in \mathrm{Pol}(T_1)$ not almost unary and such that there exists a 2-element f-wild subset of N. Then $\langle f \rangle_{T_1} = \mathrm{Pol}(T_1)$.*

2.4 Summary and a nice picture

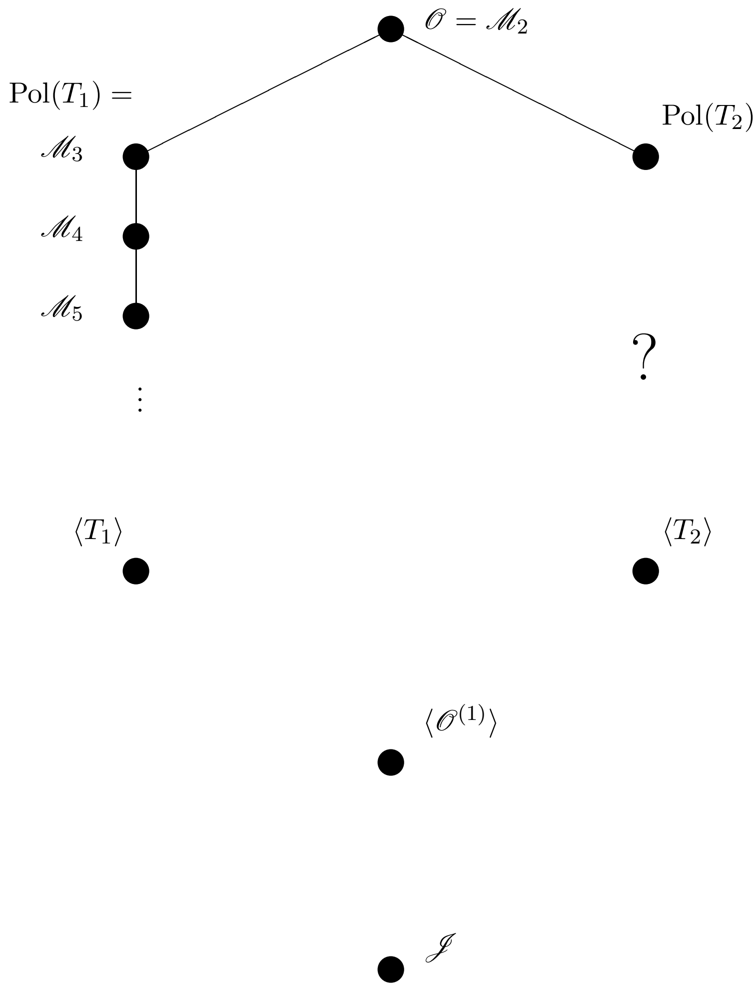
We summarize the main results of this chapter: For the interval of clones containing the almost unary functions we have $[\mathscr{U}, \mathscr{O}] = \{\mathscr{M}_2, \mathscr{M}_3, \dots, \mathscr{U}\}$, where the $\mathscr{M}_n = \langle \{m_2^n\} \cup \mathscr{U} \rangle = \langle \{m_2^n, p_\Delta\} \cup \mathscr{O}^{(1)} \rangle$ are all finitely generated over $\mathscr{O}^{(1)}$. Alternatively, $\mathscr{M}_n$ can be described as the $\leq$-downward closure of $\langle \{m_2^n\} \cup \mathscr{O}^{(1)} \rangle$. The interval is a chain: $\mathscr{M}_2 = \mathscr{O}^{(1)} \supsetneqq \mathscr{M}_3 = \mathrm{Pol}(T_1) \supsetneqq \mathscr{M}_4 \supsetneqq \dots$ and $\bigcap_{n \geq 2} \mathscr{M}_n = \mathscr{U}$. Together with the fact that $\mathscr{M}_{n+1} = \mathrm{Pol}(\mathscr{U}^{(n)})$ for all $n \geq 1$ this yields that $\mathscr{U}$ an example of a clone $\mathscr{C}$ for which the chain $\mathrm{Pol}(\mathscr{C}^{(1)}) \supseteq \mathrm{Pol}(\mathscr{C}^{(2)}) \supseteq \dots \supseteq \mathscr{C}$ is unrefinable and collapses nowhere. $\mathscr{U}$ is a so-called binary clone, that is, $\langle \mathscr{U}^{(2)} \rangle = \mathscr{U}$.

The $\mathscr{M}_n$ have the property that $\mathscr{M}_n^{(k)} = \mathscr{U}^{(k)}$ whenever $1 \leq k < n$. Furthermore, $\mathscr{M}_n^{(k)} \supsetneqq \mathscr{M}_m^{(k)}$ whenever $m > n \geq 2$ and $k \geq n$. Consequently, for each $k \geq 1$ there exist exactly k different k-ary parts of clones of the interval $[\mathscr{U}, \mathscr{O}]$.

Using wildness, a notion which completely determines a function modulo $\mathscr{U}$, it is possible to calculate for all $2 \leq k \leq n$ that $\mathscr{M}_n^k = \langle \{m_k^n\} \cup \mathscr{U} \rangle = \mathscr{M}_{\lceil \frac{n}{k-1} \rceil}$. In general, if one knows the wild subsets of $\{1,\dots,n\}$ of a function $f \in \mathscr{O}^{(n)}$, he can draw certain conclusions about where to find the clone $\langle \{f\} \cup \mathscr{U} \rangle$ in the chain.

On countable X, if we equip $\mathscr{O}$ with the natural topology, then the sets T_1 and $\mathrm{Pol}(T_1)$ are Borel sets of low complexity, as opposed to the sets T_2 and $\mathrm{Pol}(T_2)$ which have been shown by M. Goldstern to be non-analytic. In fact, with the results of this chapter, all clones above the almost unary functions can be shown to be Borel.

If X is countably infinite or weakly compact, we can draw the situation we ran into like this.



$$[\langle T_1 \rangle, \mathscr{O}] = \{\langle T_1 \rangle, \dots, \mathscr{M}_3, \mathscr{M}_2\}$$

Chapter 3

Maximal clones on uncountable sets that include all permutations

We first determine the maximal clones on a set X of infinite regular cardinality κ which contain all permutations but not all unary functions, extending a result of L. Heindorf for countably infinite X. If κ is countably infinite or weakly compact, this yields a list of all maximal clones containing the permutations since in that case the maximal clones above the unary functions are known. We then generalize a result of G. Gavrilov to obtain on all infinite X a list of all maximal submonoids of the monoid of unary functions which contain the permutations.

3.1 Background and the results

3.1.1 Clones containing the bijections

Although the clone lattice on an infinite base set X need not be dually atomic by a result of M. Goldstern and S. Shelah [GS04], the sublattice of $Cl(X)$ of clones containing the set $\mathscr{S}$ of all permutations of X is dually atomic since $\mathscr{O}$ is finitely generated over $\mathscr{S}$: Call a set $A \subseteq X$ *large* iff $|A| = |X| = \kappa$ and *small* otherwise. Moreover, A is *co-large* iff $X \setminus A$ is large, and *co-small* iff $X \setminus A$ is small. Set

$$\mathscr{I} = \{f \in \mathscr{O}^{(1)} : f \text{ is injective and } f[X] \text{ is co-large}\}$$

and

$$\mathscr{J} = \{g \in \mathscr{O}^{(1)} : g^{-1}[y] \text{ is large for all } y \in X\}.$$

It is readily verified that for arbitrary fixed $f \in \mathscr{I}$ and $g \in \mathscr{J}$ we have

$$\mathscr{I} = \{\alpha \circ f : \alpha \in \mathscr{S}\} \text{ and } \mathscr{J} = \{\alpha \circ g \circ \beta : \alpha, \beta \in \mathscr{S}\}.$$

Moreover,

$$\mathscr{O}^{(1)} = \{j \circ i : j \in \mathscr{J}, i \in \mathscr{I}\}.$$

Together with the well-known fact that $\mathscr{O}^{(1)} \cup \{p\}$ generates $\mathscr{O}$ for any binary injection p we conclude that $\mathscr{O}$ is generated by $\mathscr{S} \cup \{p, f, g\}$. Hence Zorn's lemma implies that the interval $[\mathscr{S}, \mathscr{O}]$ is dually atomic.

We will determine all maximal clones $\mathscr{C}$ on a base set of regular cardinality for which $\mathscr{S} \subseteq \mathscr{C}$ but not $\mathscr{O}^{(1)} \subseteq \mathscr{C}$. This has already been done for countable base sets by L. Heindorf in the article [Hei02] using the following concept: Let $\rho \subseteq X^J$ be a relation on X indexed by J and let $f \in \mathscr{O}^{(n)}$. We say that f *preserves* ρ iff for all $r^1 = (r^1_i : i \in J), \ldots, r^n = (r^n_i : i \in J)$ in ρ we have $(f(r^1_i, \ldots, r^n_i) : i \in J) \in \rho$. We define the clone of *polymorphisms* $\mathrm{Pol}(\rho)$ of $\rho \subseteq X^J$ to consist exactly of the functions in $\mathscr{O}$ preserving ρ. In particular, if $\rho \subseteq X^{X^k}$ is a set of k-ary functions, then the polymorphisms of ρ are exactly those $f \in \mathscr{O}^{(n)}$ for which the composite $f(g_1, \ldots, g_n) \in \rho$ whenever $g_1, \ldots, g_n \in \rho$. It is obvious that since clones are closed under composition we have $\mathscr{C} \subseteq \mathrm{Pol}(\mathscr{C}^{(n)})$ for any clone $\mathscr{C}$ and for all $n \geq 1$, where $\mathscr{C}^{(n)} = \mathscr{C} \cap \mathscr{O}^{(n)}$. Moreover, $\mathrm{Pol}(\mathscr{C}^{(n)})^{(n)} = \mathscr{C}^{(n)}$. Therefore if $\mathscr{C}$ is a maximal clone such that $\mathscr{S} \subseteq \mathscr{C}^{(1)} \subsetneqq \mathscr{O}^{(1)}$, then $\mathscr{C} \subseteq \mathrm{Pol}(\mathscr{C}^{(1)}) \subsetneqq \mathscr{O}$ holds. Hence $\mathscr{C} = \mathrm{Pol}(\mathscr{C}^{(1)})$ by the maximality of $\mathscr{C}$. We conclude that all maximal clones with $\mathscr{S} \subseteq \mathscr{C}^{(1)} \subsetneqq \mathscr{O}^{(1)}$ are of the form $\mathrm{Pol}(\mathscr{G})$ where $\mathscr{S} \subseteq \mathscr{G} \subsetneqq \mathscr{O}^{(1)}$ is a *submonoid* of $\mathscr{O}^{(1)}$, that is, a set of unary functions closed under composition and containing the identity map.

Theorem 70 (L. Heindorf). *Let X be a countably infinite set. The maximal clones over X which contain all bijections but not all unary functions are exactly those of the form $\mathrm{Pol}(\mathscr{G})$, where $\mathscr{G} \in \{\mathscr{A}, \mathscr{B}, \mathscr{D}, \mathscr{E}, \mathscr{F}\} \cup \{\mathscr{G}_n : 1 \leq n < \aleph_0\}$ is one of the following submonoids of $\mathscr{O}^{(1)}$:*

1. $\mathscr{A} = \{f \in \mathscr{O}^{(1)} : f^{-1}[\{y\}]$ *is finite for almost all* $y \in X\}$
2. $\mathscr{B} = \{f \in \mathscr{O}^{(1)} : f^{-1}[\{y\}]$ *is finite for all* $y \in X\}$
3. $\mathscr{D} = \{f \in \mathscr{O}^{(1)} : f$ *is almost injective or not almost surjective*$\}$
4. $\mathscr{E} = \{f \in \mathscr{O}^{(1)} : f$ *is almost surjective*$\}$
5. $\mathscr{F} = \{f \in \mathscr{O}^{(1)} : f$ *is almost surjective or constant*$\}$
6. $\mathscr{G}_n = \{f \in \mathscr{O}^{(1)} :$ *if* $A \subseteq X$ *has cardinality* n *then* $|X \setminus f[X \setminus A]| \geq n\}$

Consequently the number of such clones is countably infinite.

In the theorem, "almost all" means "all but finitely many", "almost injective" means that there exists a finite subset A of X such that the restriction of f to $X \setminus A$ is injective, and "almost surjective" means that the range of f is co-finite.

The restriction in the theorem to clones which do not contain all unary functions is not important since G. Gavrilov showed the following in [Gav65].

Theorem 71 (G. Gavrilov). *If X is countably infinite, then there exist exactly two maximal clones which contain $\mathscr{O}^{(1)}$.*

The two results imply that the number of maximal clones containing the permutations is countably infinite on a countably infinite base set.

We now turn to base sets of any infinite cardinality. A property $P(y)$ holds for *almost all* $y \in X$ iff the set of all elements for which the property does not hold is small. For $\lambda \leq \kappa$ a cardinal define a unary function f to be λ-*surjective* iff $|X \setminus f[X]| < \lambda$. Instead of κ-surjective we also say *almost surjective*; this means that the range of f is co-small. f is λ-*injective* iff $|\{x \in X : \exists y \neq x\,(f(x) = f(y))\}| < \lambda$. For $\lambda = 1$ or infinite, this is the case iff there exists a set $A \subseteq X$ such that $|A| < \lambda$ and such that the restriction of f to the complement of A is injective. *Almost injective* means κ-injective.

We are going to prove

Theorem 72. *Let X be a set of regular cardinality κ. The maximal clones over X which contain all bijections but not all unary functions are exactly those of the form* $\mathrm{Pol}(\mathscr{G})$*, where $\mathscr{G} \in \{\mathscr{A}, \mathscr{B}, \mathscr{E}, \mathscr{F}\} \cup \{\mathscr{G}_\lambda : 1 \leq \lambda \leq \kappa,\ \lambda \text{ a cardinal}\}$ is one of the following submonoids of $\mathscr{O}^{(1)}$:*

1. $\mathscr{A} = \{f \in \mathscr{O}^{(1)} : f^{-1}[\{y\}]$ *is small for almost all* $y \in X\}$
2. $\mathscr{B} = \{f \in \mathscr{O}^{(1)} : f^{-1}[\{y\}]$ *is small for all* $y \in X\}$
3. $\mathscr{E} = \{f \in \mathscr{O}^{(1)} : f$ *is almost surjective*$\}$
4. $\mathscr{F} = \{f \in \mathscr{O}^{(1)} : f$ *is almost surjective or constant*$\}$
5. $\mathscr{G}_\lambda = \{f \in \mathscr{O}^{(1)} :$ *if* $A \subseteq X$ *has cardinality* λ *then* $|X \setminus f[X \setminus A]| \geq \lambda\}$

Corollary 73. *Let X be a set of regular cardinality $\kappa = \aleph_\alpha$. Then there exist* $\max(|\alpha|, \aleph_0)$ *maximal clones on X which contain all bijections but not all unary functions.*

Unfortunately, we do not know the maximal clones above $\mathscr{O}^{(1)}$ on all regular cardinals; however we do on some. Let κ be a cardinal. The *partition symbol* $\kappa \to (\kappa)^2_2$ means: Whenever the edges of a complete graph with κ vertices are colored with 2

colors, then there is a complete subgraph with κ vertices, all of whose edges have the same color. An uncountable κ for which the partition relation $\kappa \to (\kappa)^2_2$ holds is called *weakly compact*. For example, the well-known theorem of F. Ramsey says that the defined partition relation holds for $\aleph_0$: If G is a complete countably infinite graph and we color its edges with two colors, then there is an infinite complete subgraph of G on which the coloring is constant.

M. Goldstern and S. Shelah [GS02] extended G. Gavrilov's result on maximal clones containing $\mathscr{O}^{(1)}$ to weakly compact cardinals.

Theorem 74 (M. Goldstern and S. Shelah). *If $\kappa = |X|$ is a weakly compact cardinal, then there exist exactly two maximal clones on X which contain $\mathscr{O}^{(1)}$.*

Hence in the case of a weakly compact base set we know all maximal clones containing the permutations. It is a fact that weakly compact cardinals κ satisfy $\kappa = \aleph_\kappa$. Thus Corollary 73 and Theorem 74 imply

Corollary 75. *Let X be a set of weakly compact cardinality κ. Then there exist κ maximal clones which contain all bijections.*

Unfortunately things are not always that easy.

Theorem 76 (M. Goldstern and S. Shelah [GS02]). *For many regular cardinalities of X, in particular for all successors of regulars, there exist $2^{2^{|X|}}$ maximal clones which contain $\mathscr{O}^{(1)}$.*

It is interesting that whereas above $\mathscr{O}^{(1)}$ the number of maximal clones varies heavily with the partition properties of the underlying base set (2 for weakly compact cardinals, 2^{2^κ} for many others), the number of maximal clones above the permutations but not above $\mathscr{O}^{(1)}$ is a monotone function of κ and always relatively small ($\leq \kappa$).

3.1.2 Maximal submonoids of $\mathscr{O}^{(1)}$

Not all monoids appearing in Theorem 72 are maximal submonoids of $\mathscr{O}^{(1)}$. More surprisingly, there exist maximal submonoids of $\mathscr{O}^{(1)}$ above the permutations whose polymorphism clone is not maximal. Observe that submonoids of $\mathscr{O}^{(1)}$ are simply *unary clones*, that is clones consisting only of essentially unary functions, and that the lattice of monoids which contain the permutations is dually atomic by the argument we have seen before.

Theorem 77 (G. Gavrilov [Gav65]). *On a countably infinite base set X the maximal submonoids of $\mathscr{O}^{(1)}$ containing the permutations are precisely the monoids $\mathscr{A}$, $\mathscr{D}$, $\mathscr{G}_1$, $\mathscr{M}$ and $\mathscr{N}$, where*

$$\mathscr{M} = \{f \in \mathscr{O}^{(1)} : f \textit{ is surjective or not injective}\}$$

and

$$\mathscr{N} = \{f \in \mathscr{O}^{(1)} : f \textit{ is almost surjective or not almost injective}\}.$$

We will generalize Theorem 77 to arbitrary infinite sets in the last section, obtaining

Theorem 78. *Let X be an infinite set. If X has regular cardinality, then the maximal submonoids of $\mathscr{O}^{(1)}$ which contain the permutations are exactly the monoid $\mathscr{A}$ and the monoids $\mathscr{G}_\lambda$ and $\mathscr{M}_\lambda$ for $\lambda = 1$ and $\aleph_0 \leq \lambda \leq \kappa$, λ a cardinal, where*

$$\mathscr{M}_\lambda = \{f \in \mathscr{O}^{(1)} : f \textit{ is } \lambda\textit{-surjective or not } \lambda\textit{-injective}\}.$$

If X has singular cardinality, then the same is true with the monoid $\mathscr{A}$ replaced by

$$\mathscr{A}' = \{f \in \mathscr{O}^{(1)} : \exists \lambda < \kappa \ (\, |f^{-1}[\{x\}]| \leq \lambda \textit{ for almost all } x \in X\,)\,\}.$$

Corollary 79. *On a set X of infinite cardinality $\aleph_\alpha$ there exist $2\,|\alpha| + 5$ maximal submonoids of $\mathscr{O}^{(1)}$ that contain the permutations. Hence the smallest cardinality on which there are infinitely many such monoids is $\aleph_\omega$.*

Observe that the statement about singular cardinals in Theorem 78 differs only slightly from the corresponding one for regulars. We do not know whether Theorem 72 can be generalized to singulars, but in our proof we do use the regularity condition.

3.1.3 Where has $\mathscr{D}$ gone?

One might ask why in the general Theorems 72 and 78 there is no monoid $\mathscr{D}$ as in Theorems 70 and 77. The answer to that question is the following: Define for $\lambda = 1$ and for all $\aleph_0 \leq \lambda \leq \kappa$ monoids

$$\delta(\lambda) = \{f \in \mathscr{O}^{(1)} : f \text{ is } \lambda\text{-injective or not } \lambda\text{-surjective}\}$$

(this definition is due to I. Rosenberg [Ros74]). Then we have

Lemma 80. *$\delta(\lambda) = \mathscr{G}_\lambda$ for $\lambda = 1$ and $\aleph_0 \leq \lambda \leq \kappa$. In particular, $\mathscr{D} = \delta(\kappa) = \mathscr{G}_\kappa$.*

Proof. Note that for $\lambda = 1$, λ-injective simply means injective and λ-surjective means surjective. The lemma is easily verified for that case, and we prove it for λ infinite.
Assuming $f \in \delta(\lambda)$ we show $f \in \mathscr{G}_\lambda$. It is clear that if f is not λ-surjective, then $f \in \mathscr{G}_\lambda$. So assume f is λ-surjective; then by the definition of $\delta(\lambda)$, f is λ-injective. Now let $A \subseteq X$ be an arbitrary set of size λ. Assume towards contradiction that $|X \setminus f[X \setminus A]| < \lambda$. Then two things can happen: If $|f[A] \cap f[X \setminus A]| \geq \lambda$, then $|\{x \in X : \exists y \neq x\,(f(x) = f(y))\}| \geq |\{x \in A : \exists y \in X \setminus A\,(f(x) = f(y))\}| \geq \lambda$, contradicting the λ-injectivity of f. Otherwise, A is mapped onto a set of size smaller than λ, again in contradiction to f being λ-injective.
To see the other inclusion, take any $f \notin \delta(\lambda)$. Then f is not λ-injective; thus we can find $A \subseteq X$ of size λ such that $f[X] = f[X \setminus A]$. But then $|X \setminus f[X \setminus A]| = |X \setminus f[X]| < \lambda$ as f is λ-surjective. Hence, $f \notin \mathscr{G}_\lambda$. ☐

Before we start with the proofs we fix some notation.

3.1.4 Notation

For a set of functions $\mathscr{F}$ we shall denote the smallest clone containing $\mathscr{F}$ by $\langle \mathscr{F} \rangle$. We call the projections which every clone contains π_i^n where $n \geq 1$ and $1 \leq i \leq n$. We write n_f for the arity of a function $f \in \mathscr{O}$ whenever that arity has not been given another name. If $a \in X^n$ is an n-tuple and $1 \leq k \leq n$ we write a_k for the k-th component of a. The image of a set $A \subseteq X^n$ under a function $f \in \mathscr{O}^{(n)}$ we denote by $f[A]$. Similarly we write $f^{-1}[A]$ for the preimage of $A \subseteq X$ under f. If $A = \{c\}$ is a singleton we cut short and write $f^{-1}[c]$ rather than $f^{-1}[\{c\}]$. Occasionally we shall denote the constant function with value $c \in X$ also by c. Whenever we identify X with its cardinality we let $<$ and $\leq$ refer to the canonical well-order on X.

3.2 The proof of Theorem 72

In this section we are going to prove Theorem 72; it will be the direct consequence of Propositions 82, 84, 85, 90, 91, 95, 96, 97, and 101. The first part of the proof (Section 3.2.1) is not much more than a translation of L. Heindorf's paper [Hei02] to arbitrary regular cardinals; the reader familiar with that article should not be surprised to find the same constructions here. In Section 3.2.2 we generalize a completeness criterion due to G. Gavrilov from countable sets to the uncountable to finish the proof.

3.2.1 The core of the proof

We start with a general observation which will be useful.

Lemma 81. *Let $\mathscr{G}$ be a proper submonoid of $\mathscr{O}^{(1)}$ such that $\langle \mathrm{Pol}(\mathscr{G}) \cup \{h\} \rangle = \mathscr{O}$ for all unary $h \notin \mathscr{G}$. Then* $\mathrm{Pol}(\mathscr{G})$ *is maximal.*

Proof. Let $f \notin \mathrm{Pol}(\mathscr{G})$ be given. Then there exist $h_1, \ldots, h_{n_f} \in \mathscr{G}$ such that $h = f(h_1, \ldots, h_{n_f}) \notin \mathscr{G}$. Now $h \in \langle \mathscr{G} \cup \{f\} \rangle \subseteq \langle \mathrm{Pol}(\mathscr{G}) \cup \{f\} \rangle$ and $\langle \mathrm{Pol}(\mathscr{G}) \cup \{h\} \rangle = \mathscr{O}$ by assumption so that we conclude $\langle \mathrm{Pol}(\mathscr{G}) \cup \{f\} \rangle = \mathscr{O}$. □

The monoids $\mathscr{A}$ and $\mathscr{B}$

Proposition 82. *The clones* $\mathrm{Pol}(\mathscr{A})$ *and* $\mathrm{Pol}(\mathscr{B})$ *are maximal.*

Proof. The maximality of $\mathrm{Pol}(\mathscr{A})$ has been proved in [Gav65] for the countable case and in [Ros74] (Proposition 4.1) for arbitrary infinite sets.

For the maximality of $\mathrm{Pol}(\mathscr{B})$, let a unary $h \notin \mathscr{B}$ be given; by Lemma 81, it suffices to show $\langle \mathrm{Pol}(\mathscr{G}) \cup \{h\} \rangle = \mathscr{O}$. By the definition of $\mathscr{B}$ there exists $c \in X$ such that the preimage $Y = h^{-1}[c]$ is large. Choose any injection $g : X \to Y$; then $h \circ g(x) = c$ for all $x \in X$.

Now let $f \in \mathscr{O}^{(n)}$ be an arbitrary function and consider $\tilde{f} \in \mathscr{O}^{n+1}$ defined by

$$\tilde{f}(x_1, \ldots, x_n, y) = \begin{cases} f(x_1, \ldots, x_n) & , y = c \\ y & , y \neq c \end{cases}.$$

We claim that $\tilde{f} \in \mathrm{Pol}(\mathscr{B})$. For let $\alpha_1, \ldots, \alpha_n, \beta \in \mathscr{B}$ and $d \in X$ be given. If $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta)(x) = d$, then by the definition of $\tilde{f}$ either $\beta(x) = c$ and $f(\alpha_1(x), \ldots, \alpha_n(x)) = d$ or $\beta(x) \neq c$ and $\beta(x) = d$. But since $\beta \in \mathscr{B}$, the set of all $x \in X$ such that $\beta(x) = c$ or $\beta(x) = d$ is small. Hence $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta)^{-1}[d]$ is small and so $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta) \in \mathscr{B}$.

Now to finish the proof it is enough to observe that $f(x_1, \ldots, x_n) = \tilde{f}(x_1, \ldots, x_n, c) = \tilde{f}(x_1, \ldots, x_n, h \circ g(x_1)) \in \langle \mathrm{Pol}(\mathscr{B}) \cup \{h\} \rangle$. □

We will prove now that $\mathscr{B}$ is the only proper submonoid of $\mathscr{A}$ whose Pol is maximal. We start with a lemma.

Lemma 83. *If $f \notin \mathrm{Pol}(\mathscr{A})$, then there exist $\alpha_1, \ldots, \alpha_{n_f} \in \mathscr{O}^{(1)}$ constant or injective such that $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{A}$.*

Proof. Since $f \notin \mathrm{Pol}(\mathscr{A})$, there exist $\beta_1, \dots, \beta_{n_f} \in \mathscr{A}$ such that $f(\beta_1, \dots, \beta_{n_f}) \notin \mathscr{A}$. We will use induction over n_f. If $n_f = 1$, then $f \notin \mathrm{Pol}(\mathscr{A})^{(1)} = \mathscr{A}$ so that $f(\pi_1^1) = f \notin \mathscr{A}$ which proves the assertion for that case. Now assume the lemma holds for all functions of arity at most $n_f - 1$. Define for $1 \leq i \leq n_f$ sets $B_i = \{y \in X : \beta_i^{-1}[y] \text{ is large}\}$. By definition of $\mathscr{A}$, all B_i are small. Set

$$\Gamma = (\beta_1, \dots, \beta_{n_f})[X] \setminus \prod_{1 \leq i \leq n_f} B_i \subseteq X^{n_f}$$

Claim. There exists a large set $D \subseteq X$ such that $f^{-1}[d] \cap \Gamma$ is large for all $d \in D$.
To prove the claim, set $D = \{d \in X : f(\beta_1, \dots, \beta_{n_f})^{-1}[d] \text{ large}\} \setminus f[\prod_{1 \leq i \leq n_f} B_i]$. The set D is large as $f(\beta_1, \dots, \beta_{n_f}) \notin \mathscr{A}$ and as $\prod_{1 \leq i \leq n_f} B_i$ is small. Define $A_d = (f(\beta_1, \dots, \beta_{n_f}))^{-1}[d]$ for each $d \in D$. Then $(\beta_1, \dots, \beta_{n_f})[A_d] \subseteq \Gamma$ is large for all $d \in D$. Indeed, assume to the contrary that there exists $d \in D$ such that $(\beta_1, \dots, \beta_{n_f})[A_d]$ is small; then, since $|X| = \kappa$ is regular, there is an $x \in (\beta_1, \dots, \beta_{n_f})[A_d]$ so that $(\beta_1, \dots, \beta_{n_f})^{-1}[x]$ is large. But then we would have $x \in \prod_{1 \leq i \leq n_f} B_i$, in contradiction to the assumption that $d \notin f[\prod_{1 \leq i \leq n_f} B_i]$. This proves the claim since $f^{-1}[d] \cap \Gamma = (\beta_1, \dots, \beta_{n_f})[A_d]$ is large for every $d \in D$.

Setting $H_b^i = \{x \in X^{n_f} : x_i = b\}$ for all $1 \leq i \leq n_f$ and all $b \in X$, we can write Γ as follows:

$$\Gamma = (\bigcup_{i=1}^{n_f} \bigcup_{b \in B_i} \Gamma \cap H_b^i) \cup (\Gamma \setminus \Delta),$$

where $\Delta = \bigcup_{i=1}^{n_f} \bigcup_{b \in B_i} H_b^i$. Since κ is regular and the union consists only of a small number of sets, we have that either there exist $1 \leq i \leq n_f$ and some $b \in B_i$ such that $f^{-1}[d] \cap \Gamma \cap H_b^i$ is large for a large set of $d \in D$, or $f^{-1}[d] \cap \Gamma \setminus \Delta$ is large for a large set of $d \in D$. We distinguish the two cases:
Case 1. There exist $1 \leq i \leq n_f$ and $b \in B_i$ such that $f^{-1}[d] \cap \Gamma \cap H_b^i$ is large for many $d \in D$; say without loss of generality $i = n_f$. Then $f(\beta_1, \dots, \beta_{n_f - 1}, b) \notin \mathscr{A}$. By induction hypothesis, there exist $\alpha_1, \dots, \alpha_{n_f - 1}$ injective or constant such that $f(\alpha_1, \dots, \alpha_{n_f - 1}, b) \notin \mathscr{A}$. Setting $\alpha_{n_f}(x) = b$ for all $x \in X$ proves the lemma.
Case 2. $f^{-1}[d] \cap \Delta$ is large for many $d \in D$. Observe that for all $a \in X$ and all $1 \leq i \leq n_f$, $\Delta \cap H_a^i$ is small, for otherwise $\beta_i^{-1}[a]$ would be large and thus $a \in B_i$, contradiction. Set

$$C = \{c \in X : f^{-1}[c] \cap \Delta \text{ large}\}.$$

By the assumption for this case, C is large. Now fix any $g : X \to C$ such that $g^{-1}[c]$ is large for all $c \in C$. We define a function $\alpha : X \to \Delta$ such that $f \circ \alpha = g$; moreover, $\alpha_i = \pi_i^{n_f} \circ \alpha$ will be injective, $1 \leq i \leq n_f$. Identify X with its cardinality κ. Then all

α_i are injective iff $\alpha_i(x) \neq \alpha_i(y)$ for all $y < x$ and all $1 \leq i \leq n_f$. This is the case iff

$$(\alpha_1, \ldots, \alpha_{n_f})(x) \in \Delta \setminus \bigcup_{y<x} \bigcup_{i=1}^{n_f} H^i_{\alpha_i(y)}.$$

Using transfinite induction on κ, we define $(\alpha_1, \ldots, \alpha_{n_f})$ by picking

$$(\alpha_1, \ldots, \alpha_{n_f})(x) \in (f^{-1}[g(x)] \cap \Delta) \setminus \bigcup_{y<x} \bigcup_{i=1}^{n_f} H^i_{\alpha_i(y)}.$$

This is possible as $f^{-1}[g(x)] \cap \Delta$ is large for all $x \in X$ whereas $\Delta \cap \bigcup_{y<x} \bigcup_{i=1}^{n_f} H^i_{\alpha_i(y)}$ is small. Clearly $f(\alpha_1, \ldots, \alpha_{n_f}) = g \notin \mathscr{A}$ and the proof of the lemma is complete. $\square$

Proposition 84. *Let $\mathscr{G} \subseteq \mathscr{A}$ be a submonoid of $\mathscr{O}^{(1)}$ which contains all permutations. Then either $\mathscr{G} \subseteq \mathscr{B}$ or $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{A})$.*

Proof. Assume $\mathscr{G} \nsubseteq \mathscr{B}$; we show $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{A})$. Observe first that for all co-large $A \subseteq X$ and all $a \in X$ there exists $g \in \mathscr{G}$ such that $g[A] = \{a\}$. Indeed, choose any $h \in \mathscr{G} \setminus \mathscr{B}$. There exists $y \in X$ such that $h^{-1}[y]$ is large. Choose bijections $\alpha, \beta \in \mathscr{S}$ with the property that $\alpha[A] \subseteq h^{-1}[y]$ and that $\beta(y) = a$. Then $g = \beta \circ h \circ \alpha$ has the desired property.
Now let $f \notin \mathrm{Pol}(\mathscr{A})$ be arbitrary; we show $f \notin \mathrm{Pol}(\mathscr{G})$. By the preceding lemma there exist $\alpha_1, \ldots, \alpha_{n_f}$ constant or injective such that $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{A}$. Choose a large and co-large $A \subseteq X$ such that $f(\alpha_1, \ldots, \alpha_{n_f})^{-1}[x] \cap A$ is large for a large set of $x \in X$. We modify the α_i to $\gamma_i \in \mathscr{G}$ in such a way that $\alpha_i \restriction_A = \gamma_i \restriction_A$ for $1 \leq i \leq n_f$: If α_i is injective, then we can choose γ_i to be a bijection. If α_i is constant, then γ_i is delivered by the observation we just made. Thus, as $f(\alpha_1, \ldots, \alpha_{n_f}) \restriction_A = f(\gamma_1, \ldots, \gamma_{n_f}) \restriction_A$ we have $f(\gamma_1, \ldots, \gamma_{n_f}) \notin \mathscr{A} \supseteq \mathscr{G}$. $\square$

Proposition 85. *Let $\mathscr{G} \subseteq \mathscr{B}$ be a submonoid of $\mathscr{O}^{(1)}$ which contains all permutations. Then $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{B})$.*

Proof. Let $f \notin \mathrm{Pol}(\mathscr{B})$ be arbitrary. We show $f \notin \mathrm{Pol}(\mathscr{G})$. There are $\beta_1, \ldots, \beta_{n_f} \in \mathscr{B}$ such that there exists $c \in X$ with the property that $f(\beta_1, \ldots, \beta_{n_f})^{-1}[c]$ is large. Define $\Gamma = (\beta_1, \ldots, \beta_{n_f})[X]$. Then since $\beta_i \in \mathscr{B}$, $H^i_a \cap \Gamma$ is small for all $1 \leq i \leq n_f$ and all $a \in X$, where $H^i_a = \{x \in X^{n_f} : x_i = a\}$. Moreover, $f^{-1}[c] \cap \Gamma$ is large. Just like at the end of the proof of Lemma 83, we can construct injective $\alpha_1, \ldots, \alpha_{n_f}$ such that $f(\alpha_1, \ldots, \alpha_{n_f})$ is constant with value c. Choose $A \subseteq X$ large and co-large and bijections $\gamma_1, \ldots, \gamma_{n_f}$ such that $\gamma_i \restriction_A = \alpha_i \restriction_A$ for $1 \leq i \leq n_f$. Then, being constant on A, $f(\gamma_1, \ldots, \gamma_{n_f}) \notin \mathscr{B} \supseteq \mathscr{G}$. Thus, $f \notin \mathrm{Pol}(\mathscr{G})$. $\square$

Generous functions

We now turn to monoids $\mathscr{G} \supseteq \mathscr{S}$ which are not submonoids of $\mathscr{A}$. Our first goal is Proposition 90, in which we give a positive description of such monoids.

Definition 86. A function $f \in \mathscr{O}^{(1)}$ is called *generous* iff $f^{-1}[y]$ is either large or empty for all $y \in X$.

Notation 87. Let $0 \leq \lambda \leq \kappa$ be a cardinal. We denote by $\mathscr{I}_\lambda$ the set of all generous functions f with the property that $|X \setminus f[X]| = \lambda$.

Lemma 88. *1. If $g \in \mathscr{O}^{(1)}$ is generous, then $f \circ g$ is generous for all $f \in \mathscr{O}^{(1)}$.*

2. $\mathscr{I}_\lambda$ is a subsemigroup and $\mathscr{I}_\lambda \cup \mathscr{S}$ a submonoid of $\mathscr{O}^{(1)}$ for all $\lambda \leq \kappa$.

3. If $\lambda < \kappa$ and $f, g \in \mathscr{I}_\lambda$, then there exist $\alpha, \beta \in \mathscr{S}$ such that $f = \alpha \circ g \circ \beta$.

4. $\mathscr{I}_\kappa$ contains all generous functions with small range, in particular the constant functions.

5. If $g \in \mathscr{I}_\kappa$ has large range, then $\langle \mathscr{S} \cup \{g\} \rangle \supseteq \mathscr{I}_\kappa$.

Proof. (1) and (4) are obvious. For (2), let $f, g \in \mathscr{I}_\lambda$; we want to show that $f \circ g \in \mathscr{I}_\lambda$. By (1) $f \circ g$ is generous, so it remains to show that $|X \setminus f \circ g[X]| = \lambda$. We distinguish two cases: If $\lambda = \kappa$, then we have $\kappa \geq |X \setminus f \circ g[X]| \geq |X \setminus f[X]| = \kappa$ and so we are finished. Otherwise we claim $f[X] = f \circ g[X]$. Indeed, if $y \in f[X]$, then $f^{-1}[y]$ is large so that $g[X] \cap f^{-1}[y] \neq \emptyset$ as $\lambda < \kappa$. Hence, $y \in f \circ g[X]$. Thus $f[X] \subseteq f \circ g[X]$ and the other inclusion is obvious.
We prove (3). Write $f[X] = \{c_i\}_{i \in \kappa}$ and $g[X] = \{d_i\}_{i \in \kappa}$. Set further $C_i = f^{-1}[c_i]$ and $D_i = g^{-1}[d_i]$ and let β_i be bijections from C_i onto D_i, $i \in \kappa$. Then $\beta = \bigcup_{i \in \kappa} \beta_i$ is a bijection on X. Define the function α by $\alpha(d_i) = c_i$ for all $i < \kappa$ and extend α to X by an arbitrary bijection from $X \setminus g[X]$ onto $X \setminus f[X]$. It is readily verified that $f = \alpha \circ g \circ \beta$.
To prove (5), let an arbitrary $f \in \mathscr{I}_\kappa$ be given; we show $f \in \langle \mathscr{S} \cup \{g\} \rangle$. Select any bijection γ with the property that $g \circ \gamma(x) = g \circ \gamma(y)$ implies $f(x) = f(y)$ for all $x, y \in X$. This is possible since g has large range and since both f and g are generous. Choose another bijection β such that for all $x, y \in X$ we have $g \circ \beta \circ g \circ \gamma(x) = g \circ \beta \circ g \circ \gamma(y)$ iff $f(x) = f(y)$. Then it is clear that there is a bijection α satisfying $f(x) = \alpha \circ g \circ \beta \circ g \circ \gamma(x)$ for all $x \in X$. ☐

Lemma 89. *If $g \notin \mathscr{A}$, then there exists $\alpha \in \mathscr{S}$ such that the function $g \circ \alpha \circ g$ is generous and has large range.*

Proof. There exists a large set $A \subseteq X$ such that $g^{-1}[a]$ is large for all $a \in A$. Set $E = g[X] \setminus A$ and $D = X \setminus g[X]$. Choose $B \subseteq A$ with the property that $A \setminus B$ is large and that $|B| = |g^{-1}[E]|$. Fix $a_0 \in A \setminus B$. Take any function $\gamma : D \to g^{-1}[A]$ making $g \circ \gamma$ injective. γ exists as A is large. We want $\alpha \in \mathscr{O}^{(1)}$ to satisfy the following properties:

(i) E shall be mapped injectively on a co-large part of $g^{-1}[a_0]$.

(ii) B shall be mapped bijectively onto $g^{-1}[E]$

(iii) $\alpha \restriction_D = \gamma$

Since E, B and D are disjoint, we can indeed choose an injective partial function $\widetilde{\alpha}$ defined on $E \cup B \cup D$ which satisfies (i)-(iii). Because $X \setminus (E \cup B \cup D) \supseteq A \setminus B$ the domain of $\widetilde{\alpha}$ is co-large. Its range is also co-large as at least a large subset of $g^{-1}[a_o]$ is not in the range. Hence we can extend $\widetilde{\alpha}$ to $\alpha \in \mathscr{S}$. We claim that α has the asserted properties. Clearly $g \circ \alpha \circ g[X] \subseteq g[X] = A \cup E$; we show that $(g \circ \alpha \circ g)^{-1}[y]$ is large for all $y \in A \cup E$. Indeed, if $y \in E$, then $(g \circ \alpha)^{-1}[y] \subseteq B \subseteq A$. Thus, the preimage of y under $g \circ \alpha \circ g$ is large. If $y \in A$, then $g^{-1}[y]$ is large and so is $g^{-1}[y] \setminus \gamma[D]$. Thus, $(g \circ \alpha)^{-1}[y] \setminus D$ is large as well. Hence, $(g \circ \alpha \circ g)^{-1}[y]$ is large which we wanted to show. □

Proposition 90. *Let $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ be a monoid containing all bijections. Then either $\mathscr{G} \subseteq \mathscr{A}$ or there exists a cardinal $\lambda \leq \kappa$ such that $\mathscr{I}_\lambda \subseteq \mathscr{G}$.*

Proof. This is an immediate consequence of Lemmas 88 and 89. □

The preceding proposition implies that when considering submonoids $\mathscr{G}$ of $\mathscr{O}^{(1)}$ which contain the permutations, we can from now on assume that $\mathscr{I}_\lambda \subseteq \mathscr{G}$ for some λ, since we already treated the case $\mathscr{G} \subseteq \mathscr{A}$. We distinguish three cases corresponding to the minimal λ with the property that $\mathscr{I}_\lambda \subseteq \mathscr{G}$: $\lambda = \kappa$, $0 < \lambda < \kappa$ and $\lambda = 0$.

The case $\lambda = \kappa$

Recall that $\mathscr{G}_\kappa$ consists of all functions $f \in \mathscr{O}^{(1)}$ with the property that whenever A is a large set then $f[X \setminus A]$ is co-large. Remember also that this is equivalent to f being either not almost surjective or almost injective.

Proposition 91. *1. $\mathrm{Pol}(\mathscr{G}_\kappa)$ is a maximal clone.*

2. If $\mathscr{G}$ is a submonoid of $\mathscr{O}^{(1)}$ such that $\mathscr{S} \cup \mathscr{I}_\kappa \subseteq \mathscr{G}$ and such that $\mathscr{I}_\lambda \subseteq \mathscr{G}$ for no $\lambda < \kappa$, then $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{G}_\kappa)$.

Proof. (1) We will prove this together with the maximality of the other $\mathrm{Pol}(\mathscr{G}_\lambda)$ in Proposition 95.
(2) Assume $f \notin \mathrm{Pol}(\mathscr{G}_\kappa)$; we show $f \notin \mathrm{Pol}(\mathscr{G})$. Take $\alpha_1, \ldots, \alpha_{n_f} \in \mathscr{G}_\kappa$ such that $g = f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{G}_\kappa$, that is, g is almost surjective but not almost injective. Choose a co-large set $A \subseteq X$ such that $g[A] = g[X]$ is co-small. Because $\alpha_i \in \mathscr{G}_\kappa$, $\alpha_i[A]$ is co-large for $1 \leq i \leq n_f$. Now fix for all $a \in A$ a large set B_a such that $A \cap B_a = \{a\}$, $B_a \cap B_{a'} = \emptyset$ whenever $a \neq a'$ and such that $X = \bigcup_{a \in A} B_a$. This is possible since A is co-large. Define for $1 \leq i \leq n_f$ functions $\beta_i \in \mathscr{O}^{(1)}$ by $\beta_i(x) = \alpha_i(a)$ whenever $x \in B_a$. It is clear that all β_i are generous. Also, since $\beta_i[X] = \alpha_i[A]$ is co-large we have $\beta_i \in \mathscr{I}_\kappa \subseteq \mathscr{G}$ for all $1 \leq i \leq n_f$. The function $f(\beta_1, \ldots, \beta_{n_f})$ is generous since it is constant on every B_a. Now $f(\beta_1, \ldots, \beta_{n_f})[X] \supseteq f(\beta_1, \ldots, \beta_{n_f})[A] = f(\alpha_1, \ldots, \alpha_{n_f})[A] = f(\alpha_1, \ldots, \alpha_{n_f})[X]$ is co-small. Hence there exists a $\lambda < \kappa$ such that $f(\beta_1, \ldots, \beta_{n_f}) \in \mathscr{I}_\lambda$, and since $\mathscr{I}_\lambda \nsubseteq \mathscr{G}$, we infer $f(\beta_1, \ldots, \beta_{n_f}) \notin \mathscr{G}$ from Lemma 88 (3) which proves $f \notin \mathrm{Pol}(\mathscr{G})$. ☐

The case $0 < \lambda < \kappa$

We shall now investigate the case where $\mathscr{G} \nsupseteq \mathscr{I}_0$ but $\mathscr{G}$ contains $\mathscr{I}_\lambda$ for some $0 < \lambda < \kappa$. We collect a couple of facts about the $\mathscr{G}_\lambda$ first. Recall that $\mathscr{G}_\lambda$ consists of those functions f for which it is true that $|X \setminus f[X \setminus A]| \geq \lambda$ whenever $A \subseteq X$ is of size λ. Recall also that for $\lambda = 1$ or infinite this is the case iff f is λ-injective or not λ-surjective.

Lemma 92. *The following statements hold for all* $1 \leq \lambda \leq \kappa$.

1. *If* $g \in \mathscr{O}^{(n)}$ *and* $|X \setminus g[X^n]| \geq \lambda$, *then* $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$.
2. $\mathscr{G}_\lambda$ *is a submonoid of* $\mathscr{O}^{(1)}$.
3. $\mathscr{G}_n \supsetneq \mathscr{G}_{n+1}$ *for all* $1 \leq n < \aleph_0$.
4. *For* $\lambda = 1$ *and for* $\lambda \geq \aleph_0$, $\mathscr{G}_\lambda$ *is a maximal submonoid of* $\mathscr{O}^{(1)}$.

Proof. (1) is obvious. For (2), let $f, g \in \mathscr{G}_\lambda$ and take an arbitrary $A \subseteq X$ with $|A| = \lambda$. Then $|X \setminus g[X \setminus A]| \geq \lambda$. Hence, $|X \setminus f \circ g[X \setminus A]| = |X \setminus f[X \setminus (X \setminus g[X \setminus A])]| \geq \lambda$ so that $f \circ g \in \mathscr{G}_\lambda$. It is clear that the identity map is an element of $\mathscr{G}_\lambda$ since it is injective. We prove (3). Observe first that the inclusion $X \setminus f[X \setminus (A \cup B)] \subseteq (X \setminus f[X \setminus A]) \cup f[B]$ holds for all $A, B \subseteq X$ and all $f \in \mathscr{O}^{(1)}$. Now let $f \in \mathscr{G}_{n+1}$ for some $1 \leq n < \aleph_0$. Take an arbitrary n-element subset A of X. Choose any $a \notin A$. Then $n + 1 \leq |X \setminus f[X \setminus (A \cup \{a\})]| \leq |(X \setminus f[X \setminus A])| + |f[\{a\}]|$ and so $n \leq |(X \setminus f[X \setminus A])|$. This proves $f \in \mathscr{G}_n$. It is obvious that $\mathscr{G}_n \neq \mathscr{G}_{n+1}$.
The proof of (4) can be found in [Ros74] (Proposition 5.2). ☐

Lemma 93. *Let $1 \leq \lambda \leq \kappa$. If $h \notin \mathscr{G}_\lambda$, then there exist a $\lambda_0 < \lambda$ such that $\langle \mathscr{I}_\lambda \cup \mathscr{S} \cup \{h\}\rangle \supseteq \mathscr{I}_{\lambda_0}$. In particular, $\langle \mathscr{G}_\lambda \cup \{h\}\rangle \supseteq \mathscr{I}_{\lambda_0}$.*

Proof. There exists $A \subseteq X$, $|A| = \lambda$ such that $|X \setminus h[X \setminus A]| < \lambda$. Set $\lambda_0 = |X \setminus h[X \setminus A]|$. Choose a generous function g with $g[X] = X \setminus A$. Then $g \in \mathscr{I}_\lambda$ since $|X \setminus g[X]| = |A| = \lambda$; thus, $h \circ g \in \langle \mathscr{I}_\lambda \cup \{h\}\rangle$. On the other hand, $h \circ g \in \mathscr{I}_{\lambda_0}$ and hence $\langle \mathscr{I}_\lambda \cup \mathscr{S} \cup \{h\}\rangle \supseteq \mathscr{I}_{\lambda_0}$ by Lemma 88 (3). The second statement is a direct consequence of the inclusion $\mathscr{G}_\lambda \supseteq \mathscr{I}_\lambda \cup \mathscr{S}$. □

Lemma 94. *Let $B \subseteq X$, $|B| = \lambda_0 < \lambda \leq \kappa$, and let $g \in \mathscr{O}^{(2)}$ such that g maps $(X \setminus B)^2$ bijectively onto X and such that $|g[B \times X] \cup g[X \times B]| < \kappa$. Then $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$.*

Proof. Let $\alpha, \beta \in \mathscr{G}_\lambda$ be given, and take an arbitrary $A \subseteq X$ of size λ. We have to show $|X \setminus g(\alpha, \beta)[X \setminus A]| \geq \lambda$. For $C = X \setminus \alpha[X \setminus A]$ we have $|C| \geq \lambda$. Thus, there exists some $c \in C \setminus B$. Obviously, $g(\alpha, \beta)[X \setminus A] \subseteq g[(X \setminus \{c\}) \times X]$. But the conditions on g yield that $g[(X \setminus \{c\}) \times X]$ and $g[\{c\} \times (X \setminus B)] \setminus (g[X \times B] \cup g[B \times X])$ are disjoint. Since $|g[\{c\} \times (X \setminus B)]| = \kappa$ and $|g[X \times B] \cup g[B \times X]| < \kappa$, this implies that $g(\alpha, \beta)$ misses κ values on $X \setminus A$ and hence, $g(\alpha, \beta) \in \mathscr{G}_\lambda$ and $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$. □

Proposition 95. *1. $\mathrm{Pol}(\mathscr{G}_\lambda)$ is a maximal clone for all $1 \leq \lambda \leq \kappa$.*

2. Let $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ be a monoid containing all bijections as well as some $\mathscr{I}_\lambda$, where $0 \leq \lambda < \kappa$, and let λ be minimal with this property. If $\lambda > 0$, then $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{G}_\lambda)$.

Proof. (1) We show $\langle \mathrm{Pol}(\mathscr{G}_\lambda) \cup \{h\}\rangle = \mathscr{O}$ for an arbitrary $h \in \mathscr{O}^{(1)} \setminus \mathscr{G}_\lambda$. By Lemma 93, there exists $\lambda_0 < \lambda$ such that $\mathscr{I}_{\lambda_0} \subseteq \langle \mathscr{G}_\lambda \cup \{h\}\rangle$. Now choose B and $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$ as in Lemma 94. Consider $\alpha : X \to (X \setminus B)^2$ such that α takes every value twice. Clearly, $\alpha_1 = \pi^2_1 \circ \alpha$ and $\alpha_2 = \pi^2_2 \circ \alpha$ are elements of $\mathscr{I}_{\lambda_0}$. The function $p = g(\alpha_1, \alpha_2) = g \circ \alpha$ maps X onto X and takes every value twice as well. Therefore we can find a co-large set A such that $p[A] = X$. Now fix a mapping $q : X \to A$ so that $p \circ q$ is the identity map on X. Let an arbitrary $f \in \mathscr{O}$ be given. Then $q \circ f[X^{n_f}] \subseteq A$ is co-large which immediately implies $q \circ f \in \mathrm{Pol}(\mathscr{G}_\lambda)$. But then $f = p \circ (q \circ f) = f \in \langle \mathrm{Pol}(\mathscr{G}_\lambda) \cup \{h\}\rangle$ and so $\langle \mathrm{Pol}(\mathscr{G}_\lambda) \cup \{h\}\rangle = \mathscr{O}$ as f was arbitrary.
(2) First we claim that $\mathscr{G} \subseteq \mathscr{G}_\lambda$. Indeed, assume there exists $h \in \mathscr{G} \setminus \mathscr{G}_\lambda$. Then, as $\mathscr{I}_\lambda \cup \mathscr{S} \subseteq \mathscr{G}$, by Lemma 93 there exists $\lambda_0 < \lambda$ such that $\mathscr{I}_{\lambda_0} \subseteq \mathscr{G}$, in contradiction to the minimality of λ.
Now let $f \notin \mathrm{Pol}(\mathscr{G}_\lambda)$ be arbitrary; we prove $f \notin \mathrm{Pol}(\mathscr{G})$. There exist $\alpha_1, \ldots, \alpha_{n_f} \in \mathscr{G}_\lambda$ such that $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{G}_\lambda$. That is, there exists $A \subseteq X$ of size λ with the property that $|X \setminus f[\Gamma]| < \lambda$, where $\Gamma = \{(\alpha_1(x), \ldots, \alpha_{n_f}(x)) : x \in X \setminus A\}$. Since $\alpha_i \in \mathscr{G}_\lambda$,

$1 \leq i \leq n_f$, for each i there exists a set $B_i \subseteq X$, $|B_i| = \lambda$, such that $\alpha_i[X \setminus A] \cap B_i = \emptyset$. Then $\Gamma \subseteq \Delta = (X \setminus B_1) \times \ldots \times (X \setminus B_{n_f})$. Choose $\beta : X \to \Delta$ onto and generous. Clearly $\beta_i = \pi_i^{n_f} \circ \beta \in \mathscr{I}_\lambda \subseteq \mathscr{G}$ for all $1 \leq i \leq n_f$. Now for all $C \subseteq X$ of size $\lambda < \kappa$ we have that $f(\beta_1, \ldots, \beta_{n_f})[X \setminus C] = f[\Delta] \supseteq f[\Gamma]$ and so, as $|X \setminus f[\Delta]| \leq |X \setminus f[\Gamma]| < \lambda$, $f(\beta_1, \ldots, \beta_{n_f}) \notin \mathscr{G}_\lambda \supseteq \mathscr{G}$. Hence, $f \notin \mathrm{Pol}(\mathscr{G})$. $\square$

The case $\lambda = 0$ and $\mathscr{G} \subseteq \mathscr{F}$

In the following proposition we treat the case where $\mathscr{I}_0 \subseteq \mathscr{G} \subseteq \mathscr{E} \subseteq \mathscr{F}$. Recall that $\mathscr{E}$ consists of those functions which are almost surjective (that is, κ-surjective).

Proposition 96. *1.* $\mathrm{Pol}(\mathscr{E})$ *is a maximal clone.*

2. If $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ *is a monoid containing all bijections as well as* $\mathscr{I}_0$*, and if* $\mathscr{G} \subseteq \mathscr{E}$*, then* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{E})$.

Proof. (1) We prove that for any unary $h \notin \mathscr{E}$ we have $\langle \mathrm{Pol}(\mathscr{E}) \cup \{h\} \rangle = \mathscr{O}$. By definition $h[X]$ is co-large, so we can fix $A \subseteq X$ large and co-large such that $A \cap h[X] = \emptyset$. Choose any $g \in \mathscr{O}^{(1)}$ which maps A onto X and which is constantly $0 \in X$ on $X \setminus A$. Then $g \in \mathscr{E}$ as it is onto. Moreover, $g \circ h$ is constantly 0. Now let an arbitrary $f \in \mathscr{O}^{(n)}$ be given and define a function $\tilde{f} \in \mathscr{O}^{n+1}$ by

$$\tilde{f}(x_1, \ldots, x_n, y) = \begin{cases} f(x_1, \ldots, x_n) & , y = 0 \\ y & , \text{otherwise} \end{cases}$$

Then $\tilde{f} \in \mathrm{Pol}(\mathscr{E})$. Indeed, this follows from the inclusion $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta)[X] \supseteq \beta[X] \setminus \{0\}$ for arbitrary $\alpha_1, \ldots, \alpha_n, \beta \in \mathscr{O}^{(1)}$. Now $f(x) = \tilde{f}(x, 0) = \tilde{f}(x, g \circ h(x_1))$ for all $x \in X^n$ and so $f \in \langle \mathrm{Pol}(\mathscr{E}) \cup \{h\} \rangle$.
(2) Taking an arbitrary $f \notin \mathrm{Pol}(\mathscr{E})$ we show that $f \notin \mathrm{Pol}(\mathscr{G})$. There exist $\alpha_1, \ldots, \alpha_{n_f}$ almost surjective such that $f(\alpha_1, \ldots, \alpha_{n_f})$ is not almost surjective. Consider a small set $A \subseteq X$ so that $A \cup \alpha_i[X] = X$ for all $1 \leq i \leq n_f$. Let γ be a surjection from $X \setminus A$ onto X and define for $1 \leq i \leq n_f$ functions

$$\beta_i(x) = \begin{cases} \alpha_i \circ \gamma(x) & , x \in X \setminus A \\ x & , x \in A \end{cases}$$

Clearly, all β_i are surjective and $f(\beta_1, \ldots, \beta_{n_f})[X] = f(\alpha_1, \ldots, \alpha_{n_f})[X] \cup \{f(x, \ldots, x) : x \in A\}$ is co-large. Fix any $\delta \in \mathscr{I}_0$. Obviously $\beta_i \circ \delta \in \mathscr{I}_0 \subseteq \mathscr{G}$ and also $f(\beta_1 \circ \delta, \ldots, \beta_{n_f} \circ \delta)[X]$ is co-large. Thus $f(\beta_1 \circ \delta, \ldots, \beta_{n_f} \circ \delta) \notin \mathscr{E} \supseteq \mathscr{G}$ so that we infer $f \notin \mathrm{Pol}(\mathscr{G})$. $\square$

In a next step we see what happens in the case $\mathscr{I}_0 \subseteq \mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G} \nsubseteq \mathscr{E}$. $\mathscr{F}$ is the set of those functions which are almost surjective or constant.

Proposition 97. *1.* $\mathrm{Pol}(\mathscr{F})$ *is a maximal clone.*

2. If $\mathscr{G} \subseteq \mathscr{F}$ *is a monoid which contains* $\mathscr{I}_0$ *as well as all bijections, and if* $\mathscr{G} \nsubseteq \mathscr{E}$*, then* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{F})$.

Proof. (1) can be found in [Ros74] (Proposition 3.1).
For (2), let $f \notin \mathrm{Pol}(\mathscr{F})$ and fix $\alpha_1, \ldots, \alpha_{n_f} \in \mathscr{F}$ satisfying $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{F}$. Since $\mathscr{G} \nsubseteq \mathscr{E}$ but $\mathscr{G} \subseteq \mathscr{F}$, $\mathscr{G}$ must contain a constant function, and hence all constant functions as $\mathscr{S} \subseteq \mathscr{G}$. For those of the α_i which are not constant we construct β_i as in the proof of the preceding proposition, and for the constant ones we set $\beta_i = \alpha_i$. Observe that it is impossible that all α_i are constant. Choosing any $\delta \in \mathscr{I}_0$ we obtain that for all $1 \leq i \leq n_f$, $\beta_i \circ \delta$ is either constant or an element of $\mathscr{I}_0$, and hence in either case an element of $\mathscr{G}$. But as in the preceding proof, $f(\beta_1 \circ \delta, \ldots, \beta_{n_f} \circ \delta) \notin \mathscr{F} \supseteq \mathscr{G}$ so that $f \notin \mathrm{Pol}(\mathscr{G})$. □

The case $\lambda = 0$ and $\mathscr{G} \nsubseteq \mathscr{F}$

To conclude, we consider submonoids $\mathscr{G}$ of $\mathscr{O}^{(1)}$ which contain the bijections as well as $\mathscr{I}_0$, but which are not submonoids of $\mathscr{F}$. It turns out that the polymorphism clones of such monoids are never maximal. We start with a simple fact about such monoids.

Lemma 98. *Let* $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ *be a monoid containing* $\mathscr{S} \cup \mathscr{I}_0$ *such that* $\mathscr{G} \nsubseteq \mathscr{F}$*. Then* $\chi = \{\rho \in \mathscr{O}^{(1)} : |\rho[X]| = 2$ *and* ρ *is generous*$\} \subseteq \mathscr{G}$.

Proof. Let $f \in \mathscr{G} \setminus \mathscr{F}$. Since f is not constant there exist $a \neq b$ in the range of f. Let $s : X \setminus f[X] \to X$ be onto and generous and define $g \in \mathscr{O}^{(1)}$ by

$$g(x) = \begin{cases} s(x) & , x \notin f[X] \\ a & , x = a \\ b & , \text{otherwise} \end{cases}$$

Then $g \in \mathscr{I}_0 \subseteq \mathscr{G}$ and so $g \circ f \circ g \in \mathscr{G}$. On the other hand, $g \circ f \circ g \in \chi$ which proves the lemma since obviously any function of χ together with the permutations generate all of χ.

□

To prove that the remaining monoids do not yield maximal clones via Pol, we are going to generalize the following completeness criterion due to G. Gavrilov [Gav65] (Lemma 31 on page 51) for countable base sets.

Lemma 99 (G. Gavrilov). *Let X be countably infinite. If $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ is a monoid containing $\mathscr{S} \cup \mathscr{I}_0 \cup \chi$, and if $\mathscr{H} \subseteq \mathscr{O}$ is a set of functions such that $\langle \mathscr{O}^{(1)} \cup \mathscr{H} \rangle = \mathscr{O}$, then $\langle \mathscr{G} \cup \mathscr{H} \rangle = \mathscr{O}$.*

So we claim

Proposition 100. *Lemma 99 holds on all base sets of infinite regular cardinality.*

It follows immediately that $\mathrm{Pol}(\mathscr{G})$ is not maximal for the remaining monoids $\mathscr{G}$.

Proposition 101. *If $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ is a monoid such that $\mathscr{S} \cup \mathscr{I}_0 \subseteq \mathscr{G}$ and such that $\mathscr{G} \nsubseteq \mathscr{F}$, then* $\mathrm{Pol}(\mathscr{G})$ *is not maximal.*

Proof. We have just seen that $\chi \subseteq \mathscr{G}$ so we can apply Proposition 100. Suppose towards contradiction that $\mathrm{Pol}(\mathscr{G})$ is maximal. Since $\mathrm{Pol}(\mathscr{G})^{(1)} = \mathscr{G} \subsetneqq \mathscr{O}^{(1)}$ we have $\langle \mathscr{O}^{(1)} \cup \mathrm{Pol}(\mathscr{G}) \rangle = \mathscr{O}$. But then setting $\mathscr{H} = \mathrm{Pol}(\mathscr{G})$ in the lemma yields that $\langle \mathscr{G} \cup \mathrm{Pol}(\mathscr{G}) \rangle = \mathscr{O}$, which is impossible as $\langle \mathscr{G} \cup \mathrm{Pol}(\mathscr{G}) \rangle = \mathrm{Pol}(\mathscr{G}) \neq \mathscr{O}$, contradiction. ☐

3.2.2 The proof of Proposition 100.

Notation 102. We set $\mathscr{L} = \langle \chi \cup \mathscr{I}_0 \cup \mathscr{S} \rangle$. Moreover, we write Const for the set of all constant functions.

The following description of $\mathscr{L}$ is readily verified.

Lemma 103. $\mathscr{L} = \mathrm{Const} \cup \chi \cup \mathscr{I}_0 \cup \mathscr{S}$. *In words, $\mathscr{L}$ consists exactly of the bijections as well as of all generous functions which are either onto or take at most two values.*

Lemma 104. *Let $u \in \mathscr{O}^{(1)}$ be injective and not almost surjective. Then $\langle \{u\} \cup \mathscr{I}_0 \rangle \supseteq \mathscr{O}^{(1)}$. In particular, $\langle \{u\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Proof. Let an arbitrary $f \in \mathscr{O}^{(1)}$ be given. Take any $s : X \setminus u[X] \to X$ which is generous and onto. Now define $g \in \mathscr{O}^{(1)}$ by

$$g(x) = \begin{cases} f(u^{-1}(x)) & , x \in u[X] \\ s(x) & , \text{otherwise} \end{cases}$$

Since $g \restriction_{X \setminus u[X]} = s$ we have $g \in \mathscr{I}_0$. Clearly, $f = g \circ u \in \langle \{u\} \cup \mathscr{I}_0 \rangle$. ☐

Definition 105. A function $f(x_1, \ldots, x_n) \in \mathscr{O}^{(n)}$ is *almost unary* iff there exist a function $F : X \to \mathscr{P}(X)$ and some $1 \leq k \leq n$ such that $F(x)$ is small for all $x \in X$ and such that for all $(x_1, \ldots, x_n) \in X^n$ we have $f(x_1, \ldots, x_n) \in F(x_k)$. We denote the set of all almost unary functions by $\mathscr{U}$.

It is easy to see that on a base set of regular cardinality, $\mathscr{U}$ is a clone which contains $\mathscr{O}^{(1)}$. See [Pin04b] for a list of all clones above $\mathscr{U}$; there are countably many, so in particular $\mathscr{U}$ is not maximal. The reason for us to consider almost unary functions is the following lemma.

Lemma 106. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$ be any function which is not almost unary. Then $\langle \{f\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Observe that this lemma implies that $\mathrm{Pol}(\mathscr{G}) \subseteq \mathscr{U}$ for all proper submonoids $\mathscr{G}$ of $\mathscr{O}^{(1)}$ which contain $\mathscr{L}$ and that we can therefore conclude directly that these polymorphism clones are not maximal. We will now prove Lemma 106 by showing that $\mathscr{L}$ together with a not almost unary f generate a function u as in Lemma 104. We start by observing that $\mathscr{L}$ and f generate functions of arbitrary range.

Lemma 107. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$. Then there exists a unary $g \in \langle \{f\} \cup \mathscr{L} \rangle$ such that the range of g is large and co-large.*

Proof. We distinguish two cases.
Case 1. For all $1 \leq i \leq n$ and all $c \in X$ it is true that $f[X^{i-1} \times \{c\} \times X^{n-i}]$ is co-small. Then consider an arbitrary large and co-large $A \subseteq X$. Set $\Gamma = f^{-1}[X \setminus A] \subseteq X^n$ and let $\alpha : X \to \Gamma$ be onto. By the assumption for this case, $f[X^{i-1} \times \{c\} \times X^{n-i}] \setminus A$ is still large for all $1 \leq i \leq n$ and all $c \in X$. Thus the components $\alpha_i = \pi_i^n \circ \alpha$ are generous and onto; hence, $\alpha_i \in \mathscr{I}_0 \subseteq \mathscr{L}$ for all $1 \leq i \leq n$. But now $f(\alpha_1, \ldots, \alpha_n)[X] = f[X^n] \setminus A$ is large and co-large so that it suffices to set $g = f \circ \alpha$.
Case 2. There exists $1 \leq i \leq n$ and $c \in X$ such that $f[X^{i-1} \times \{c\} \times X^{n-i}]$ is co-large, say without loss of generality $i = 1$. Since $f \notin \mathscr{U}$, there exists $d \in X$ satisfying that $f[\{d\} \times X^{n-1}]$ is large. Choose $\Gamma \subseteq X^{n-1}$ large and co-large such that $f[\{d\} \times \Gamma]$ is large and such that $f[\{c\} \times X^{n-1}] \cup f[\{d\} \times \Gamma]$ is still co-large. Take moreover $\alpha_2, \ldots, \alpha_n \in \mathscr{I}_0$ so that $(\alpha_2, \ldots, \alpha_n)[X] = X^{n-1}$. Now we define $\alpha_1 \in \mathscr{O}^{(1)}$ by

$$\alpha_1(x) = \begin{cases} d & , (\alpha_2, \ldots, \alpha_n)(x) \in \Gamma \\ c & , \text{otherwise.} \end{cases}$$

Clearly, $\alpha_1 \in \chi \subseteq \mathscr{L}$. Now it is enough to set $g = f(\alpha_1, \ldots, \alpha_n)$ and observe that $g[X] = f[\{c\} \times (X^{n-1} \setminus \Gamma)] \cup f[\{d\} \times \Gamma]$ is large and co-large. ∎

Lemma 108. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$. Then for all $A \subseteq X$ there exists $h \in \langle \{f\} \cup \mathscr{L} \rangle$ with $h[X] = A$.*

Proof. By Lemma 107 there exists $g \in \langle \{f\} \cup \mathscr{L} \rangle$ having a large and co-large range. Now taking any $\delta \in \mathscr{I}_0 \subseteq \mathscr{L}$ with $\delta[g[X]] = A$ and setting $h = \delta \circ g$ proves the assertion. □

Lemma 109. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$, then $\langle \{f\} \cup \mathscr{L} \rangle$ contains all generous functions.*

Proof. Let any generous $g \in \mathscr{O}^{(1)}$ be given and take with the help of the preceding lemma $h \in \langle \{f\} \cup \mathscr{L} \rangle$ with $h[X] = g[X]$. By setting $h' = h \circ \delta$, where $\delta \in \mathscr{I}_0 \subseteq \mathscr{L}$ is arbitrary, we obtain a generous function with the same property. Now it is clear that there exists a bijection $\sigma \in \mathscr{S} \subseteq \mathscr{L}$ such that $g = h' \circ \sigma$. □

Now that we know that we have all generous functions we want to make them injective. We start by reducing the class of functions f under consideration.

Lemma 110. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$ is so that for all $1 \leq i \leq n$ and for all $a, b \in X$ the set of all tuples $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X^{n-1}$ with $f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \neq f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$ is small, then $\langle \{f\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Proof. Since $f \notin \mathscr{U}$ we can for every $1 \leq i \leq n$ choose $c_i \in X$ such that $f[X^{i-1} \times \{c_i\} \times X^{n-i}]$ is large. Choose moreover for every $1 \leq i \leq n$ large sets $A_i \subseteq f[X^{i-1} \times \{c_i\} \times X^{n-i}]$ such that $\bigcup_{i=1}^{n} A_i$ is co-large and such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Write each A_i as a disjoint union of many large sets: $A_i = \bigcup_{x \in X} A_i^x$. Let $\lhd$ be any well-order of X^n of type κ. Define $\Gamma \subseteq X^n$ by $x \in \Gamma$ iff there exists $1 \leq i \leq n$ such that $f(x) \in A_i^{x_i}$ and whenever $y \lhd x$ and $y \in \Gamma$ then $f(x) \neq f(y)$. Observe that the latter condition ensures that $f \restriction_\Gamma$ is injective.

Now observe that for all $1 \leq i \leq n$, all $c \in X$ and all large $B \subseteq A_i$ we have that $f[X^{i-1} \times \{c\} \times X^{n-i}] \cap B$ is large. Indeed, say without loss of generality $i = 1$ and set $D = \{(x_2, \ldots, x_n) : f(c, x_2, \ldots, x_n) \neq f(c_1, x_2, \ldots, x_n)\}$. Then D is small by our assumption. Now $|f[\{c\} \times X^{n-1}] \cap B| \geq |f[\{c\} \times (X^{n-1} \setminus D)] \cap B| = |f[\{c_1\} \times (X^{n-1} \setminus D)] \cap B| = \kappa$. In particular, this observation is true for $B = A_i^c$. This implies that the set $\{x \in \Gamma : x_i = c\}$ is large for all $1 \leq i \leq n$ and all $c \in X$. Moreover, Γ itself is large.

Therefore there exists a bijection $\alpha : X \to \Gamma$. By the preceding observation, the components $\alpha_i = \pi_i^n \circ \alpha$ are onto and generous, so $\alpha_i \in \mathscr{I}_0 \subseteq \mathscr{L}$ for all $1 \leq i \leq n$. Since α is injective, $\alpha[X] = \Gamma$ and $f \restriction_\Gamma$ is injective, we have that $g = f(\alpha_1, \ldots, \alpha_n) \in \langle \{f\} \cup \mathscr{L} \rangle$ is injective. Furthermore, $g[X] = f[\Gamma] \subseteq \bigcup_{i=1}^{n} A_i$ is co-large. Whence $\mathscr{O}^{(1)} \subseteq \langle \{g\} \cup \mathscr{L} \rangle \subseteq \langle \{f\} \cup \mathscr{L} \rangle$ by Lemma 104 and we are done. □

Lemma 111. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$ is so that for all $1 \leq i \leq n$ there exist $c \in X$ and $S \subseteq X^n$ with $\pi_i^n[S] = \{c\}$ such that $f[S]$ large and such that for all $b \in X$ the set $\{x \in S : f(x) \neq f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)\}$ is small, then $\langle \{f\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Proof. Fix for every $1 \leq i \leq n$ an element $c_i \in X$ and a set $S_i \subseteq X^n$ such that $\pi_i^n[S_i] = \{c_i\}$ and such that $f[S_i]$ large and such that for all $b \in X$ the set $\{x \in S_i : f(x) \neq f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)\}$ is small. Set $A_i = f[S_i]$, $1 \leq i \leq n$. By thinning out the S_i we can assume that the A_i are disjoint and that $\bigcup_{i=1}^n A_i$ is co-large. Now one follows the proof of the preceding lemma. □

Lemma 112. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$, then there exists $g \in \langle \{f\} \cup \mathscr{L} \rangle$ having co-large range and with the property that $\{x \in X : |g^{-1}[x]| = 1\}$ is large (that is, the kernel of g has κ one-element classes).*

Proof. There is nothing to prove if f satisfies the condition of Lemma 111, so assume it does not, and let $i = 1$ witness this. Take $c \in X$ such that $f[\{c\} \times X^{n-1}]$ is large and choose $S \subseteq X^n$ such that $\pi_1^n[S] = \{c\}$, such that $f[S]$ is still large and such that $f \restriction_S$ is injective. By the lemma, there exists $b \in X$ such that $\{x \in S : f(x) \neq f(b, x_2, \ldots, x_n)\}$ is large. Thus, we can find a large $A \subseteq S$ with the property that $f[A]$ and $f[\{(b, x_2, \ldots, x_n) : x \in A\}]$ are disjoint and such that the union of these two sets is co-large. Choose now generous $\alpha_2, \ldots, \alpha_n \in \mathscr{O}^{(1)}$ such that $(c, \alpha_2, \ldots, \alpha_n)[X] = A$. Since $\langle \{f\} \cup \mathscr{L} \rangle$ contains all generous functions by Lemma 109, we have $\alpha_j \in \mathscr{L}$ for $2 \leq j \leq n$. Take a large and co-large $B \subseteq X$ such that $(c, \alpha_2, \ldots, \alpha_n) \restriction_B$ is injective. Define

$$\alpha_1(x) = \begin{cases} c & , x \in B \\ b & , \text{otherwise} \end{cases}$$

and set $g = f(\alpha_1, \ldots, \alpha_n)$. Then $g \in \langle \{f\} \cup \mathscr{L} \rangle$ as $\alpha_1 \in \chi \subseteq \mathscr{L}$. Clearly, $(\alpha_1, \ldots, \alpha_n) \restriction_B$ is injective and so is $g \restriction_B$. Since $g[B]$ and $g[X \setminus B]$ are disjoint we have that $|g^{-1}[x]| = 1$ for all $x \in g[B]$. Moreover, $g[X] \subseteq f[A] \cup f[\{(b, x_2, \ldots, x_n) : x \in A\}]$ is co-large. □

Lemma 113. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$. If $h \in \mathscr{O}^{(1)}$ is a function whose kernel has at least one large equivalence class (that is, there exists $x \in X$ with $h^{-1}[x]$ large), then $h \in \langle \{f\} \cup \mathscr{L} \rangle$.*

Proof. There exist a large $B \subseteq X$ and $b \in X$ such that $h[B] = \{b\}$. Let g be provided by the preceding lemma. With the help of permutations of the base set we can assume that $|g^{-1}[x]| = 1$ for all $x \in g[X \setminus B]$. Since the range of g is co-large we can find

$\delta : X \setminus g[X] \to X$ onto and generous. Now define $m \in \mathscr{O}^{(1)}$ by

$$m(x) = \begin{cases} \delta(x) & , x \notin g[X] \\ b & , x \in g[B] \\ h(g^{-1}(x)) & , x \in g[X \setminus B]. \end{cases}$$

Obviously $m \in \mathscr{I}_0 \subseteq \mathscr{L}$ and $h = m \circ g \in \langle \{f\} \cup \mathscr{L} \rangle$. $\square$

Having found many functions which $\langle \{f\} \cup \mathscr{L} \rangle$ must contain, we are finally ready to prove Lemma 106.

Proof of Lemma 106. There are $c_1, \ldots, c_n \in X$ such that $f[X^{i-1} \times \{c_i\} \times X^{n-i}]$ is large for $1 \leq i \leq n$. Take $B_1, \ldots, B_n$ large such that $\pi_i^n[B_i] = \{c_i\}$ for all $1 \leq i \leq n$ and with the property that $f \restriction_B$ is injective and $f[B]$ is co-large, where $B = \bigcup_{i=1}^n B_i$. Let $\alpha : X \to B$ be any bijection. Since $\alpha_i^{-1}[c_i]$ is large for every component $\alpha_i = \pi_i^n \circ \alpha$, the preceding lemma yields $\alpha_i \in \langle \{f\} \cup \mathscr{L} \rangle$ for $1 \leq i \leq n$. Whence, $g = f(\alpha_1, \ldots, \alpha_n) \in \langle \{f\} \cup \mathscr{L} \rangle$. But $g[X] = f[B]$ is co-large and g is injective by construction; thus Lemma 104 yields $\mathscr{O}^{(1)} \subseteq \langle \{g\} \cup \mathscr{L} \rangle \subseteq \langle \{f\} \cup \mathscr{L} \rangle$. $\square$

This brings us back to our original goal.

Proof of Proposition 100. Since $\langle \mathscr{O}^{(1)} \cup \mathscr{H} \rangle = \mathscr{O}$, there must exist some $f \in \mathscr{H} \setminus \mathscr{U}$. But then, since $\mathscr{G} \supseteq \mathscr{L}$, Lemma 106 implies $\langle \mathscr{G} \cup \mathscr{H} \rangle \supseteq \mathscr{O}^{(1)}$ so that we infer $\langle \mathscr{G} \cup \mathscr{H} \rangle = \mathscr{O}$. $\square$

3.3 The proof of Theorem 78

We now determine on an infinite X all maximal submonoids of $\mathscr{O}^{(1)}$ which contain the permutations, proving Theorem 78. In a first section, we present the part of the proof which works on all infinite sets; then follow one section specifically for the case of a base set of regular cardinality and another section for the singular case. Throughout all parts we will mention explicitly whenever a statement is true only on X of regular or singular cardinality, respectively.

3.3.1 The part which works for all infinite sets

Proposition 114. *$\mathscr{G}_\lambda$ is a maximal submonoid of $\mathscr{O}^{(1)}$ for $\lambda = 1$ and $\aleph_0 \leq \lambda \leq \kappa$.*

Proof. As already mentioned in Lemma 92, the maximality of the $\mathscr{G}_\lambda$ for $\lambda = 1$ or infinite has been proved in [Ros74] (Proposition 5.2). $\square$

The maximal monoids of Proposition 114 already appeared in the preceding section since they give rise to maximal clones via Pol. We shall now expose maximal monoids above the permutations which do not have this property. Recall that $\mathscr{M}_\lambda$ consists of all functions which are either λ-surjective or not λ-injective.

Proposition 115. *Let $\lambda = 1$ or $\aleph_0 \leq \lambda \leq \kappa$. Then $\mathscr{M}_\lambda$ is a maximal submonoid of $\mathscr{O}^{(1)}$.*

Proof. We show first that $\mathscr{M}_\lambda$ is closed under composition. Let therefore $f, g \in \mathscr{M}_\lambda$, that is, those functions are either λ-surjective or not λ-injective; we claim that $f \circ g$ has either of these properties. It is clear that if g is not λ-injective, then $f \circ g$ has the same property. So let g be λ-surjective. It is easy to see that if f is λ-surjective, then so is $f \circ g$. So assume finally that f is not λ-injective. We claim that $f \circ g$ is not λ-injective either. For $\lambda = 1$ this is just the statement that if f is not injective, and g is surjective, then $f \circ g$ is not injective, which is obvious. Now consider the infinite case. There exist disjoint $A, B \subseteq X$ of size λ such that $f[A] = f[B]$. Set $A' = A \cap g[X]$; A' still has size λ as g misses less than λ values. Clearly $B' = \{x \in B : \exists y \in A'(f(x) = f(y))\}$ has size λ as well and so does $B'' = B' \cap g[X]$. But now for the sets $C = g^{-1}[A']$ and $D = g^{-1}[B'']$ it is true that $|C|, |D| \geq \lambda$, $C \cap D = \emptyset$, and $f \circ g[C] = f \circ g[D]$; hence $f \circ g$ is not λ-injective.
Now we prove that $\mathscr{M}_\lambda$ is maximal in $\mathscr{O}^{(1)}$. Consider for this reason any $m \notin \mathscr{M}_\lambda$, that is, m is λ-injective and misses at least λ values. There exists $A \subseteq X$ so that $|X \setminus A| < \lambda$ and such that the restriction of m to A is injective. Take any injection $i \in \mathscr{O}^{(1)}$ with $i[X] = A$. Then $i \in \mathscr{M}_\lambda$ as i is λ-surjective. Now let $f \in \mathscr{O}^{(1)}$ be arbitrary. Define

$$g(x) = \begin{cases} f((m \circ i)^{-1}(x)) & , x \in m \circ i[X] \\ a & , \text{otherwise} \end{cases}$$

where $a \in X$ is any fixed element of X. Being constant on the complement of the range of m, g it is not λ-injective and whence an element of $\mathscr{M}_\lambda$. Therefore $f = g \circ m \circ i \in \langle \mathscr{M}_\lambda \cup \{m\} \rangle$ so that we infer $\langle \mathscr{M}_\lambda \cup \{m\} \rangle \supseteq \mathscr{O}^{(1)}$. $\square$

Lemma 116. *There are no other maximal monoids above $\mathscr{S} \cup \mathscr{I}_0$ except the $\mathscr{M}_\lambda$ ($\lambda = 1$ or $\aleph_0 \leq \lambda \leq \kappa$).*

Proof. Let $\mathscr{G} \supseteq \mathscr{I}_0 \cup \mathscr{S}$ be a submonoid of $\mathscr{O}^{(1)}$ which is not contained in any of the $\mathscr{M}_\lambda$; we prove that $\mathscr{G} = \mathscr{O}^{(1)}$. To do this, we show that $\mathscr{G}$ contains an injective function $u \in \mathscr{O}^{(1)}$ with co-large range; then the lemma follows from Lemma 104. Fix for every λ a function $m_\lambda \in \mathscr{G} \setminus \mathscr{M}_\lambda$. Since m_κ is κ-injective, there exists a cardinal $\lambda_1 < \kappa$ and a set $A_1 \subseteq X$ of size λ_1 such that the restriction of m_κ to the complement of A_1 is

injective. If λ_1 is infinite, then consider m_{λ_1}. Not being an element of $\mathscr{M}_{\lambda_1}$, m_{λ_1} misses at least λ_1 values. Hence by adjusting it with a suitable permutation we can assume that $m_{\lambda_1}[X] \subseteq X \setminus A_1$. There exists a cardinal $\lambda_2 < \lambda_1$ and a subset A_2 of X of size λ_2 such that the restriction of m_{λ_1} to the complement of A_2 is injective. Hence, writing $\lambda_0 = \kappa$ we obtain that $m_{\lambda_0} \circ m_{\lambda_1} \in \mathscr{G}$ is injective on $X \setminus A_2$ and misses κ values. We can iterate this to arrive after a finite number of steps at a set A_n of finite size λ_n such that the restriction of $m_{\lambda_0} \circ \ldots \circ m_{\lambda_{n-1}} \in \mathscr{G}$ to $X \setminus A_n$ is injective and misses κ values. Since $m_1 \notin \mathscr{M}_1$ is injective and misses at least one value we conclude that the iterate $m_1^{\lambda_n} \in \mathscr{G}$ is injective and misses at least λ_n values. Modulo permutations we may assume that $m_1^{\lambda_n}[X] \subseteq X \setminus A_n$. But now we have that $m_{\lambda_0} \circ \ldots \circ m_{\lambda_{n-1}} \circ m_1^{\lambda_n} \in \mathscr{G}$ is injective and misses κ values, implying that $\mathscr{G} = \mathscr{O}^{(1)}$. □

3.3.2 The case of a base set of regular cardinality

We now finish the proof of Theorem 78 for the case when X has regular cardinality. The proof for this case comprises Propositions 114, 115, 117 and 118.

Proposition 117. *If X is of regular cardinality, then $\mathscr{A}$ is a maximal submonoid of $\mathscr{O}^{(1)}$.*

Proof. This has been proved in [Ros74] (Proposition 4.1). □

Proposition 118. *Let X have regular cardinality. There exist no other maximal submonoids of $\mathscr{O}^{(1)}$ containing the permutations except those listed in Theorem 78 for the regular case.*

Proof. Assume that $\mathscr{G} \supseteq \mathscr{S}$ is a submonoid of $\mathscr{O}^{(1)}$ not contained in any of the monoids of the theorem; we show that $\mathscr{G} = \mathscr{O}^{(1)}$. Indeed, since $\mathscr{G} \nsubseteq \mathscr{A}$, Proposition 90 tells us that there exists a cardinal $\lambda \leq \kappa$ such that $\mathscr{I}_\lambda$ is contained in $\mathscr{G}$. Choose λ minimal with this property. If λ was greater than 0, then $\mathscr{G} \subseteq \mathscr{G}_\lambda$ for otherwise Lemma 93 would yield a contradiction to the minimality of λ. But this is impossible as we assumed that $\mathscr{G}$ is not contained in any of the $\mathscr{G}_\lambda$, so we conclude that $\lambda = 0$. Now Lemma 116 implies that $\mathscr{G} = \mathscr{O}^{(1)}$. □

3.3.3 The case of a base set of singular cardinality

The only problem with base sets of singular cardinality is that the set $\mathscr{A}$ is not closed under composition; in fact, $\langle \mathscr{A} \rangle = \mathscr{O}$. A slight adjustment of the definition of $\mathscr{A}$ works in this case. We will refer to results from preceding sections; this might look unsafe since there we restricted ourselves to base sets of regular cardinality. However, when

proving the particular results cited here we did not use the regularity of the base set. The proof of Theorem 78 for singular cardinals comprises Propositions 114, 115, 122 and 123.

Definition 119. A function $f \in \mathscr{O}^{(1)}$ is said to be *harmless* iff there exists $\lambda < \kappa$ such that the set of all $x \in X$ for which $|f^{-1}[x]| > \lambda$ is small. With this definition, $\mathscr{A}'$ as defined in Theorem 78 is the set of all harmless functions.

Lemma 120. *$\mathscr{A}'$ is a monoid and $\mathscr{A}' \subseteq \mathscr{A}$. Moreover, $\mathscr{A} = \mathscr{A}'$ iff κ is a successor cardinal.*

Proof. It is obvious that $\mathscr{A}' \subseteq \mathscr{A}$ and that $\mathscr{A} = \mathscr{A}'$ iff κ is a successor cardinal. To prove that $\mathscr{A}'$ is closed under composition, let $f, g \in \mathscr{A}'$; we show $h = f \circ g \in \mathscr{A}'$. There exist $\lambda_f, \lambda_g < \kappa$ witnessing that f and g are harmless. Set λ to be $\max(\lambda_f, \lambda_g)$; we claim that the set of $x \in X$ for which $|h^{-1}[x]| > \lambda$ is small. For if $|h^{-1}[x]| > \lambda$, then either $|g^{-1}[x]| > \lambda$ or there exists $y \in g^{-1}[x]$ such that $|f^{-1}[y]| > \lambda$. Both possibilities occur only for a small number of $x \in X$ and so h is harmless. ☐

Lemma 121. *Let X have singular cardinality. If $g \notin \mathscr{A}'$, then g together with $\mathscr{S}$ generate a function not in $\mathscr{A}$.*

Proof. Set $\lambda < \kappa$ to be the cofinality of κ. Because g is not harmless, there exist distinct sequences $(x^0_\xi)_{\xi<\lambda}, \ldots, (x^\kappa_\xi)_{\xi<\lambda}$ of distinct elements of X such that $\bigcup_{\xi<\lambda} g^{-1}[x^\zeta_\xi]$ is large for all $\zeta < \kappa$. Indeed, if $(\mu_\xi)_{\xi<\lambda}$ is any cofinal sequence of cardinalities in κ, then the fact that g is not harmless allows us to pick for every $\xi < \lambda$ an element $x^0_\xi \in X$ such that $|g^{-1}[x^0_\xi]| > \mu_\xi$; it is also no problem to choose the elements distinct. This yields the first sequence and since with every sequence we are using up only $\lambda < \kappa$ elements, the definition of harmlessness ensures that we can repeat the process κ times. By throwing away half of the sequences, we may assume that the set of all $y \in X$ which do not appear in any of the sequences is large.

There exists a permutation $\alpha \in \mathscr{S}$ such that $g \circ \alpha(x^{\zeta_1}_{\xi_1}) = g \circ \alpha(x^{\zeta_2}_{\xi_2})$ if and only if $\zeta_1 = \zeta_2$, for all $\zeta_1, \zeta_2 < \kappa$ and all $\xi_1, \xi_2 < \lambda$. For we can map every sequence $(x^\zeta_\xi)_{\xi<\lambda}$ injectively into an equivalence class of the kernel of g of size greater than λ; since there are many such classes every sequence can be assigned an own class, and we choose the classes so that a large number of classes are not hit at all. This partial injective mapping we can then extend to the permutation α as it is defined on a co-large set and has co-large range.

Set $y^\zeta = g \circ \alpha(x^\zeta_0)$ for all $\zeta < \kappa$. Then the y^ζ are pairwise distinct and for all $\zeta < \kappa$ we have that $(g \circ \alpha \circ g)^{-1}[y^\zeta] \supseteq \bigcup_{\xi<\lambda} g^{-1}[x^\zeta_\xi]$ is large. Hence, $g \circ \alpha \circ g \notin \mathscr{A}$. ☐

Proposition 122. *Let X have singular cardinality. Then $\mathscr{A}'$ is a maximal submonoid of $\mathscr{O}^{(1)}$.*

Proof. Let $g \in \mathscr{O}^{(1)} \setminus \mathscr{A}'$. We know that g together with $\mathscr{A}'$ generate a function not in $\mathscr{A}$. Then by Lemma 89, we obtain a function which is generous and has large range, call it h. Now take any $f \in \mathscr{O}^{(1)}$ such that $f \circ h[X] = X$ which is injective on $h[X]$ and constant on $X \setminus h[X]$. Then $f \in \mathscr{A}'$ and $f \circ h \in \mathscr{I}_0$. Thus, $\mathscr{I}_0 \subseteq \langle \{g\} \cup \mathscr{A}' \rangle$ and since all injections are elements of $\mathscr{A}'$ we can apply Lemma 104 to prove $\langle \{g\} \cup \mathscr{A}' \rangle \supseteq \mathscr{O}^{(1)}$. □

Proposition 123. *Let X have singular cardinality. There exist no other maximal submonoids of $\mathscr{O}^{(1)}$ containing the permutations except those listed in Theorem 78 for the singular case.*

Proof. If $\mathscr{G} \supseteq \mathscr{S}$ is a submonoid of $\mathscr{O}^{(1)}$ which is not contained in $\mathscr{A}'$, then it is not contained in $\mathscr{A}$ by Lemma 121. From this point, one can follow the proof of Proposition 118. □

Bibliography

[DR85] R. O. Davies and I. G. Rosenberg. Precomplete classes of operations on an uncountable set. *Colloq. Math.*, 50:1–12, 1985.

[Gav65] G. P. Gavrilov. On functional completeness in countable-valued logic (Russian). *Problemy Kibernetiki*, 15:5–64, 1965.

[Gol0x] M. Goldstern. Analytic clones. preprint, 200x.

[GS02] M. Goldstern and S. Shelah. Clones on regular cardinals. *Fundamenta Mathematicae*, 173(1):1–20, 2002.

[GS04] M. Goldstern and S. Shelah. Clones from creatures. *Transactions of the American Mathematical Society*, 2004. to appear.

[Hei02] L. Heindorf. The maximal clones on countable sets that include all permutations. *Algebra univers.*, 48:209–222, 2002.

[Kec95] A. Kechris. *Classical descriptive set theory.* Springer, 1995.

[Pin02] M. Pinsker. Rosenberg's characterization of maximal clones. Diploma thesis, Vienna University of Technology, 2002.

[Pin04a] M. Pinsker. The clone generated by the median functions. *Contrib. Gen. Algebra*, 15:99–108, 2004.

[Pin04b] M. Pinsker. Clones containing all almost unary functions. *Algebra Univers.*, 2004. to appear.

[Pin0x] M. Pinsker. Maximal clones on uncountable sets that include all permutations. 200x. submitted.

[PK79] R. Pöschel and L. Kalužnin. *Funktionen- und Relationenalgebren.* VEB Deutscher Verlag der Wissenschaften, 1979.

[Pos41] E. L. Post. *The Two-Valued Iterative Systems of Mathematical Logic*, volume 5 of *Ann. Math. Studies*. Princeton University Press, 1941.

[Ros70] I. G. Rosenberg. Über die funktionale Vollständigkeit in den mehrwertigen Logiken. *Rozpravy Československé Akad. věd, Ser. Math. Nat. Sci.*, 80:3–93, 1970.

[Ros74] I. G. Rosenberg. Some maximal closed classes of operations on infinite sets. *Math. Annalen*, 212:157–164, 1974.

[Ros76] I. G. Rosenberg. The set of maximal closed classes of operations on an infinite set A has cardinality $2^{2^{|A|}}$. *Arch. Math. (Basel)*, 27:561–568, 1976.

[Sze86] Á. Szendrei. *Clones in universal algebra*. Les Presses de L'Université de Montréal, 1986.

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